Surface Area and Volume – Learn it, Live it, and Apply it!

Elizabeth Denenberg

Objectives

My unit for the Delaware Teacher's Institute is a 10^{th} grade mathematics unit that focuses on teaching students to measure and understand the surface area and volume of three-dimensional figures. Students will learn to find the surface area and volume of complex three-dimensional shapes using appropriate mathematical strategies. The geometry content involves key solids such as prisms, pyramids, cones, cylinders, and spheres. It will enable them to understand the meaning of surface area and volume as well as the derivation of their formulas. Students will be able to find unknown dimensions of shapes given their properties.

In order to reach these objectives, students will start by constructing and analyzing shapes using paper and pencil first, and then transitioning to technology. It is vital to see a figure's characteristics before calculating its measurements. By using and constructing three-dimensional models, students are better able to develop spatial and proportional reasoning.[i] According the NCTM standards, all high school students should "draw and construct representations of two- and three-dimensional geometric objects using a variety of tools." This is the basis for the beginning of my unit in order to build a strong understanding of geometric shapes. My focus is on how students can learn a deep understanding of geometric measurement rather than just a basic knowledge of a shape's measurement formulas.

I am following Bloom's taxonomy in order to progress student thinking from the lowest to the highest levels. In Level I, students will define and label key two- and three-dimensional shapes. In Level II, students will classify shapes by their properties, compare shapes' characteristics and measurements, and select formulas to make calculations. In Level III, students will construct shapes and apply appropriate formulas to find surface areas and volumes. In Level IV, students will analyze the geometric formulas to justify why they work and make sense, examine the connections between surface area and volume formulas for related figures, and solve for unknown dimensions of shapes using their problem solving and reasoning skills. Finally, at the highest level of Bloom's Taxonomy, Level V, students will design their own formulas for making measurements of more complex shapes by using their knowledge of the basic three-dimensional figures. They will theorize how to solve volume and area problems involving composite shapes, shapes inscribed in other shapes, or shapes obtained from cutting pieces out of basic shapes.

Too often when students learn about surface and volume, the activity becomes a "plug and chug" event where they simply try to find the right numbers to plug into a formula given to them. These calculations are meaningless and irrelevant to them. My goal is give my students a deeper understanding of surface area and volume measurements so that students can use more complex mathematical reasoning in order to solve problems with geometric solids.

In the Math Grade-Level Expectations for the state of Delaware, there are key concepts that students must learn in grade 10 related to surface area and volume as part of Standard 3- Geometric Reasoning. As part of measurement, students must be able to "find unknown dimensions of a shape given the area, volume, or surface area." Students must also "use partitioning and formulas to find the surface area and volume of complex shapes." These are the key objectives I would like my students to achieve. Both are listed as essential concepts in the Delaware state standards. However, I don't believe that all my students have the necessary background to reach this level of understanding, so I am also focusing on 9th grade standards in order to build a stronger foundation for my students. In Standard 3 as part of measurement for 9th graders, students must "demonstrate an understanding of and apply formulas for area, surface area, and volume of geometric figures including pyramids, cones, spheres, and cylinders." In my unit, we will also be focusing on prisms. These are the three state standards that are essential in my unit.

School Background and Curriculum

I teach at Hodgson Vocational Technical High School in Newark, Delaware. Because we are a vocational school, all of our students spend part of their day in an academic setting and part of their day in a selected vocational shop. We have a wide variety of vocational options ranging from masonry, plumbing, auto-body, electrical trades, dental assisting, business tech, and cosmetology, to name a few. Geometry and measurement concepts are present in many of our shops. Often, students come to my math class and tell me how they used measurement techniques or constructed geometrical objects in their shop that day. At Hodgson, we emphasize how academics can be used in everyday life. Therefore, it is important to present problems that have real world applications. Our student motto is "Learn it, live it, and apply it." This motto is repeated every day during the morning announcements. In a setting like this, it is vital to look for opportunities to apply mathematical concepts to areas that are relevant for my students.

In the curriculum for our Integrated Mathematics II course at Hodgson, we use the Core-Plus Mathematics textbook series. Students complete a unit called Patterns in Shape in which there is a lesson on three-dimensional objects. Students learn to identify prisms, pyramids, cylinders, and cones by their properties. They learn to construct nets and draw orthographic drawings to model these solids. However, there is a not an emphasis on finding and understanding the surface area and volume of these solids.

There are homework problems in the curriculum that have students find the surface area and volume of prisms and pyramids with the same height and then compare their results. These problems are not relevant and effective for students until they have an understanding of the meaning of these measurements. When given these formulas, they usually end up plugging numbers into the variables for height, length, width, radius, base area, etc. without thinking about the meaning of these dimensions. It is necessary that students first learn where these formulas came from and why they are relevant so that they avoid making careless mathematical mistakes and implement their problem solving and reasoning skills appropriately. In order to understand formulas, they must first understand the objects that the formulas apply to.

Content

This unit concentrates on finding measurements of three-dimensional objects. However, classifying objects by their properties is also important. In order to solve problems involving surface area and volume, students must first learn about the characteristics of the objects they are measuring. For example, a person can usually recognize a cylinder, but they may not be able to mathematically justify *why* it is a cylinder. Therefore, my unit will begin with understanding the properties of shapes and then we will transition to measurement.

Two-dimensions vs. Three-dimensions

Though my unit is based on three-dimensional objects, it is important to understand certain key two-dimensional properties and their relationship to three-dimensional space. When students work in two-dimensions, they learn to measure angles and side lengths. Furthermore, they learn to calculate areas and perimeters. They learn to classify two shapes as congruent. Any two two-dimensional shapes are congruent if they have all the same corresponding angle measures and corresponding side lengths, and therefore, their areas and perimeters are equal as well. Students must be able to name and measure shapes by analyzing their properties. A correct and rigorous understanding of these concepts is essential in order to master the objectives of my unit because two-dimensional shapes represent the faces of three-dimensional shapes.

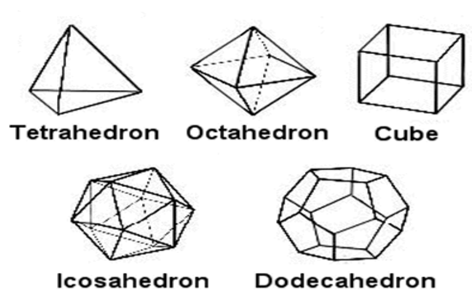
A common mistake that I hear from students is they identify a three-dimensional object by a two-dimensional name. For example, a student may be given a cube, but they call it a square. This demonstrates that the student does not understand the difference between two and three dimensional objects. A cube, for example, is a three-dimensional object whose faces are all squares. Even though the two figures, a cube and a square, have a connection, their meanings are very different. Two-dimensional objects, such as a square, lie in a given plane. There are many suitable definitions for a plane, but the key idea is that it is a flat surface. A three-dimensional object is any object such that not all of its elements lie in a given plane. In the example of the cube, each of its faces is a square, which each lie in their own plane. No two faces lie in the same plane. Therefore, the square faces are two-dimensional, but the cube itself is three-dimensional.

Polyhedra

A polyhedron is a three-dimensional solid whose faces are all polygons. A polygon is a two-dimensional figure made up of vertices and the same number of sides. Common polygons are triangles, quadrilaterals, pentagons, etc. The number of sides determines the name of the polygon. The word polyhedron comes from the Greek *poly*- meaning many, and *hedron*- meaning faces. So polyhedron translates literally to “many faces.” It is important to keep in mind that these faces must be polygons. Perhaps one of the best ways for students to understand a polyhedron is to think of counterexamples. For example, a cylinder is not a polyhedron because its top and bottom faces are circles, which is not a polygon since a circle has no side lengths or angles. Other three-dimensional counterexamples include spheres and cones.

All polyhedra have faces, vertices, and edges. An edge of a polyhedron is defined as the intersection of any two faces. A vertex is the intersection of at least three edges. The number of edges that meet at any vertex can vary, but it must be at least three. The relationship between the number of faces F, the number of vertices V, and the number of edges E, in a polyhedron is given by Euler’s Formula: $V + F = E + 2$. This formula holds true for any polyhedron. Students should be able to take any polyhedron and see that this holds by counting vertices, faces, and edges.

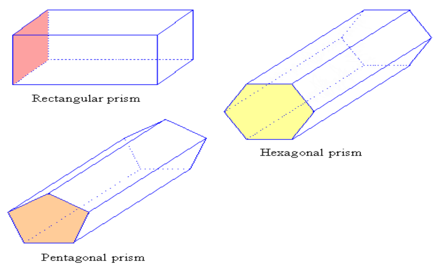
There are many ways to classify polyhedra. If all the faces are congruent regular polygons, with the same number of faces meeting at each vertex, then these polyhedral are called platonic solids. There are five platonic solids: the tetrahedron (4 triangular faces), the hexahedron (6 square faces, also known as a cube), the octahedron (8 triangular faces), the dodecahedron (12 pentagonal faces), and the icosahedron (20 triangular faces). Examples of the five platonic solids are shown below:



These figures can be used to help students understand polyhedra, identify vertices, edges, and faces, as well as an object that can verify Euler’s formula. However, because my unit is about measuring surface area and volume, we will only really concentrate on two special types of polyhedra, prisms and pyramids.

Prisms

A prism is a polyhedron that has two congruent parallel bases and whose lateral faces are all parallelograms. We name a prism by the type of polygon that is its base. For example, if the base shape of a prism is a hexagon, we call that a hexagonal prism. When calculating the surface area and volume of a prism, the key dimensions will be the dimensions of the polygonal base and the height of the prism. The height of a prism is defined as the distance between the two congruent parallel bases. Shown below are examples of three prisms that are named according to their bases.

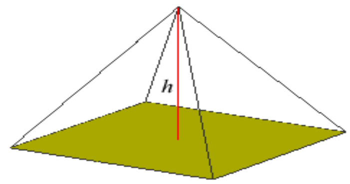


Pyramids

A pyramid is also a type of polyhedron. It is defined as a three-dimensional object that has a polygonal base whose lateral faces are triangles that meet at an apex. Similar to the prism, we name a pyramid according to its base shape. If the base of pyramid is a triangle, we call this object a triangular pyramid.

The measurements of any pyramid include the dimensions of the polygonal base, the height of the pyramid and the length of the edges of the triangular faces that are not part of the base of a pyramid. The height of a pyramid is measured by the shortest distance from the apex of the pyramid to the plane containing the base which we measure on a perpendicular line so this segment, from the apex to the base, forms a right angle with the plane containing the base. A pyramid is a right pyramid if the segment representing the height of the pyramid will lie in the center of the polygonal base.

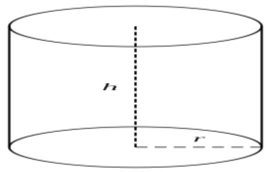
A regular pyramid is a right pyramid whose base is a regular polygon and all lateral edges are congruent. Regular pyramids have another important measurement called slant height. The slant height of a regular pyramid is the height of a triangular face of the pyramid. All lateral faces of regular pyramids are congruent isosceles triangles. The picture shown below is a regular pyramid with a square base. This object is known as a square pyramid.



The majority of the pyramids used in my unit will be regular pyramids because it allows students to make many more connections among the dimensions of a pyramid. For example, in the regular square pyramid shown above, there is a connection between the pyramid's height, slant height and edge length. This connection can be expressed by using the Pythagorean Theorem. Let *x* denote the side lengths of the square base and let *h* denote the height of the pyramid. We can construct a segment from the bottom of the height of the pyramid (located in the center of the square) to the midpoint of any side of the base. Because it is a right pyramid, this segment will have a length equal to ½ *x* and be perpendicular to the side of the square. A right triangle is now formed with legs of length *h* and ½ *x*, and a hypotenuse equal to the slant height of the pyramid, *s*. Therefore, we can calculate the slant height of the pyramid in terms of *h* and *x*, using the formula $s = \sqrt{h^2 + (\frac{1}{2}x)^2}$. There is also a right triangle formed with the slant height *s* and half an edge of the square base ½ *x*, with a hypotenuse length that is equal to the edge length of the lateral triangles of the pyramid. Let *z* denote the edge length. This triangle can be formed on any of the lateral faces of the pyramid. Therefore, $z = \sqrt{s^2 + (\frac{1}{2}x)^2}$. You can also see through this example that the edge length, *z*, is always greater than the slant height, *s*, which is always greater than the height of the pyramid, *h*.

Cylinders

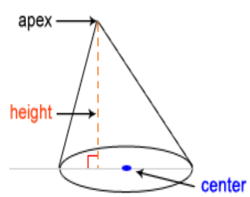
A cylinder is a solid with two congruent parallel bases that are circles. The line segment determined by the centers of the bases is called the axis of the cylinder. The cylinder is one of the most basic solids in geometry but a more mathematical definition is not often given. The lateral surface of the cylinder is the surface generated by moving a segment connecting two points on the bases, and parallel to the axis, around the bases. The cylinder itself is the solid enclosed by this surface and the two planes perpendicular to the axis. Most of the cylinders in my unit will be right cylinders. If the axis is perpendicular to the bases, then the cylinder is called a right cylinder and all points on the lateral surface are equally distanced from the axis. In a right cylinder, the angle formed between axis of the cylinder and any radius of the base will be a right angle. The cylinder shown below is an example of a right cylinder.



In a right cylinder, the axis of the cylinder is also the height of the cylinder. The height of a cylinder is the shortest distance between the two bases found by constructing a segment perpendicular to the circular bases. In an oblique cylinder, the axis is slanted and therefore is longer than the length of the height of the cylinder. In this unit, and in high school curriculum, only right cylinders are included in problems.

Cone

A cone is a solid with a circular base and a vertex, called an apex, located at a point that does not lie in the same plane as its base. It is formed by connecting the apex to every point on the perimeter of the circular base. A cone is considered a right cone if the line connecting the apex to the center of the circular base is perpendicular to the base. This line is called the axis of the cone. For a right cone, the distance of the segment is called the height of the cone. The height of a cone is measured by the shortest distance from the apex of the cone to the plane containing the base. This segment, from the apex to the base, forms a right angle with the plane containing the base. For an oblique cone, which is any cone that is not a right cone, the line segment representing the height of the cone will not pass through the center of the circular base as shown:



Sphere

A sphere is a perfectly round three-dimensional object. Similar to a circle, it is defined by its center and radius. A circle, however, is two-dimensional, meaning it only lies in a single plane. One way to see the connection between a circle and a sphere is that a sphere can be formed by rotating a circle around one of its diameters. A sphere has a center such that all points on its surface are the same distance away from the center. The distance from the center of the sphere to its surface is called the radius of the sphere.

Surface Area

The surface area of a three-dimensional object is found by finding the sum of the area of all exposed surfaces of a solid. The exposed surfaces of a solid represent the barrier between the interior of the object and the space outside the object. Because you are finding the area of a surface, this is a two-dimensional measurement. Therefore, all measurements of surface area are in square units. You can find the surface area of a polyhedron by adding up the area of all of its faces. It is important that students know how to find the area of common polygons, such as squares, parallelograms, rectangles, and triangles, because often these shapes make up the faces of polyhedron.

Nets

One of the clearest ways to understand and calculate the surface area of a solid is to use the net of the figure. A net is a two-dimensional representation of a three-dimensional object that shows its exposed faces. A net can be folded up in order to form the three-dimensional solid. You find the surface area of any solid by finding the area of its net. Helping students understand the relationship between the surface area of a three-dimensional object and the area of its two-dimensional net is vital for students' conceptual understanding of surface area.[ii]

Deriving surface area formulas

In most high school mathematics classrooms, certain surface area formulas are given to students. However, the derivation of these formulas is a key understanding in order to fully comprehend the concept of surface area. The surface area of a cube of side *s* is given by the formula $SA_{cube} = 6s^2$. This formula comes from the fact that a cube has 6 congruent square faces. Each face has a side length of s, so each square face has an area of s^2. Since surface area comes from finding the area of all exposed faces, we can add up all six individual face areas to get $s^2 + s^2 + s^2 + s^2 + s^2 + s^2 = 6s^2$. The surface area of a right cylinder of height *h* and radius *r* is given by the formula $SA_{cylinder} = 2\pi rh + 2\pi r^2$. In order to understand the derivation of this formula, it helps to analyze the net of a cylinder. The net of a cylinder consists of two congruent circles, which are the parallel bases of the solid when it is formed, and a rectangle with a height equal to the height of the cylinder. The hardest measurement to visualize is the length of the rectangle. Because the cylinder is formed by curving the rectangle so it aligns with the circle, the length of the rectangle is equal to the circumference of the circle. Therefore, we can derive the surface area of a cylinder by adding up the areas of the three shapes as shown:

$$SA_{cylinder} = Area_{circle1} + Area_{circle2} + Area_{rec\tan gle} = \pi r^2 + \pi r^2 + h \cdot (2\pi r) = 2\pi r^2 + 2\pi rh .$$

There are ways to derive many other surface area formulas of solids, but I believe it is only necessary for students to understand and practice on the main basic solids. Some surface area derivations involve very complex mathematics above the high school level. The importance is not they can derive a formula for surface area, but that they understand the logic involved in the derivation so that it deepens their understanding of the meaning of surface area.

Surface Area of complex shapes

A large part of my unit will involve finding measurements of complex shapes, meaning the shapes will involve a composite of known solids or the figure may involve missing corners or holes. It is important for students to remember that surface area only involves the exposed areas. When composite shapes are formed, sometimes the faces of the original solids are now hidden. On the other hand, there can also be newly exposed faces that must be included in the calculation of surface area.

For example, if I want to find the surface area of a composite shape where a cone is placed on top of a cylinder, with the same base, you cannot just add the surface area of the cylinder to the surface area of the cone. Once the cone is placed on top of the cylinder, the base of the cone and the top circular base of the cylinder are no longer exposed because the bases overlap and become part of the interior of the new object. The surface area of a cone is found by the formula $SA_{cone} = \pi r^2 + 2\pi rs$, where *r* represents the radius of the base and *s* is the slant height of the cone. To answer this problem, you can think about it one of two ways. You can find its surface area by adding up the surface areas of the two figures individually and subtracting the two faces that are no longer exposed. In this case, the calculation looks like $SA_{composite} = SA_{cone} + SA_{cylinder} - Area_{conebase} - Area_{cylindertop}$. The dimensions needed for this problem are the height of the cylinder *h*, the radius of the cylinder *r*, which is also the radius of the cone, and the slant height of the cone *s*. The calculation then is $SA_{composite} = (\pi r^2 + 2\pi rs) + (2\pi r^2 + 2\pi rh) - \pi r^2 - \pi r^2 = \pi r^2 + 2\pi rs + 2\pi rh$. Another plausible way to approach this problem is to simply find the area of the exposed surfaces of the composite shape. There are three exposed surfaces in the composite shape. They are the circular base of the cylinder, the lateral surface of the cylinder, and the lateral area of the cone. The top circular base of the cylinder and the base of the cone are outer faces of the original solids but are now part of the interior composite solid so they are not part of the surface area calculation.

Another type of complex shape that students will investigate is a solid generated by cutting one type of solid out of the same type or other types of solid. For example, given a cylinder with a smaller cylinder cut out of it, what is the surface area of the resulting object? This can be a relevant question since this shape is similar to pipes or tubes seen in the real world. The two cylinders will have the same height, *h*. The larger cylinder that represents the outside of the pipe will have a radius, r_1, and the smaller cylinder inside will have a radius, r_2. Once again, in order to calculate surface area, you need to determine the areas that are exposed. In this case, there is the lateral surface of the larger cylinder, $2\pi r_1 h$, and the lateral surface of the smaller cylinder, $2\pi r_2 h$, which lies in the inside of the tube shape. Then there are two congruent rings on the top and bottom of the tube that have an area equal to the difference between the area of the larger circular base and the area of the smaller circular base. The area of two rings together is $2 \cdot (\pi r_1^2 - \pi r_2^2)$. So the surface area of the tube will be $SA_{tube} = 2\pi r_1 h + 2\pi r_2 h + 2(\pi r_1^2 - \pi r_2^2)$.

Volume

The volume of a three-dimensional object is found by calculating the amount of space enclosed by a figure. Volume is measured by calculating the number of unit cubes, *1unit x1 unit x1 unit* cubes that fit into a given space. It is a three-dimensional measurement which means the units to any volume calculation will be cubic units. In our mathematics curriculum as well as the majority of mathematics curricula in the country, students are provided with the volume formulas of basic solids. On any district test, our students are given a formula sheet with the following volume formulas:

Solid	**Volume Formula**
Cube	$V = s^3$ *s = side of the cube*
Pyramid	$V = \frac{1}{3}Bh$ *B = area of base, h = height*
Cylinder	$V = \pi r^2 h$
Cone	$V = \frac{1}{3}\pi r^2 h$ *r = radius of the base circle*
Sphere	$V = \frac{4}{3}\pi r^3$ *r = radius of the sphere*
Prism	$V = Bh$ *B = area of base*

Because these formulas are provided, students become very reliant on finding the appropriate formula and plugging in numbers rather than analyzing the task at hand. The volume problems in my unit will force students to go beyond the “plug and chug” level of thinking. We will investigate the derivation of some of these formulas, calculate volumes of complex shapes, and analyze the results of inscribing solids in one another.

Learning the volume formulas

All of the volume formulas provided can be derived through mathematical reasoning. Some are easier to visualize than others. I would like students to analyze the formulas for the volume of a cube, a cylinder and a prism. All three of these formulas are written differently on the formula table, but they can actually all be derived by using the formula for the volume of a prism. The volume of a prism is given by $V = Bh$, meaning the volume of a prism is the area of the base multiplied by the height of the prism. Students can make sense of this formula if they understand that a prism can be constructed by layering “mini prisms” with the same base and a height of 1 unit. There would be a total of h layers, where h represents the height of the prism. Each layer will contain B unit cubes so the volume formula for a prism must be $V = Bh$. If the prism was a right prism, these layers would be stacked straight up, and if it was an oblique prism, the layers would be stacked on a slant. However, this would not affect the volume of the prism because you will still have same number of layers. A similar argument works for a cylinder and a cube. For a cylinder, your base shape is a circle, meaning each cylindrical layer will have πr^2 unit cubes. The height of a cylinder is given by h, making the volume formula for a cylinder $V = \pi r^2 \cdot h$. A cube is a type of prism such that the base is a square, with side lengths s, and the height of the prism is also s. Therefore, a cube can be constructed by layering square prisms with a height of 1 unit, so each layer will consist of s^2 unit cubes. There will be s layers making the volume of a cube $V = s^2 \cdot s = s^3$.

Volume of complex shapes

If you give a student a cone, cylinder, sphere, etc. with given dimensions, finding its volume is not only an easy task, but it is meaningless from both the geometric point of view and the algebraic calculation point of view. By using more complex shapes, students will be forced to think more conceptually and use mathematical reasoning to solve volume problems. Students must identify the objects and their attributes and select the right formulas corresponding to the known and unknown variables.

Inscribing three-dimensional objects

Inscribing an object means to place an object into another such that is completely enclosed. All vertices of the inscribed solid will be touching the surface of the second solid. In my unit, students will investigate the affects of inscribing different three-dimensional solids in a given three-dimensional solid. The most basic example would be to inscribe objects in a cube whose edge length is denoted by s. We can investigate at inscribing prisms, pyramids, cylinders, cones, and spheres in this cube. A key question is what is the largest of these shapes that can fit in the cube? We can define the term “largest” as having the greatest volume. Before doing this problem in three-dimensions, you could investigate this problem in two-dimensions, by looking at what shapes can be inscribed inside a square. You could inscribe two-dimensional figures such as circles, triangles, or even another square with vertices at the midpoints of each side of the square in the square. In this problem, you would compare the area of the inscribed shapes with one another and with the area of the square.

In three-dimensions, we can start by inscribing a prism in a cube. We can discount a square prism because a square prism, with side length *s* and height *s*, would fit perfectly inside the cube. Its volume would be the same as the cube. We will instead choose prisms with non-square bases. For example, a triangular prism whose base shape was an equilateral triangle. The largest triangular prism that you can fit in this cube will have a base triangle with side lengths *s* and the height of the prism will be *s*. The equilateral triangle will have an area of $\frac{\sqrt{3}}{4}s^2$, so the triangular prism will have a volume of $V = \frac{\sqrt{3}}{4}s^2 \cdot s = \frac{\sqrt{3}}{4}s^3$. Next, we can inscribe a pyramid in the cube. The largest possible pyramid that can be inscribed is a square pyramid with base edges of length *s* and a height of *s*. The volume of this pyramid will be $V = \frac{1}{3}s^2 \cdot s = \frac{1}{3}s^3$. The largest possible cylinder that can be inscribed in this cube will have a radius of ½ *s* and height of *s*. Its volume will be $V = \pi(\frac{1}{2}s)^2 \cdot s = \pi \cdot \frac{1}{4}s^2 \cdot s = \frac{\pi}{4}s^3$. Similar to the cylinder, the largest possible cone that can be inscribed in the cube will have a radius of ½ *s* and a height of *s*. The volume of this cone is $V = \frac{1}{3}\pi(\frac{1}{2}s)^2 \cdot s = \frac{1}{3}\pi \cdot \frac{1}{4}s^2 \cdot s = \frac{\pi}{12}s^3$. The last solid I will inscribe in the cube is a sphere. I have noticed in the past that students often believe that the sphere will be the largest solid that can fit in the cube, but as you will see, it is not so. The largest sphere that can fit in this cube will have a radius of ½ *s*. Its volume can be calculated as $V = \frac{4}{3}\pi(\frac{1}{2}s)^3 = \frac{4}{3}\pi \cdot \frac{1}{8}s^3 = \frac{\pi}{6}s^3$. The volume of five inscribed solids can all be written as a constant multiplied by s^3, so by looking at the value of the constants, it is easy to rank these inscribed solids from largest to smallest. It may help to write the constants as rounded decimals so that they are easy to compare. The results are shown below with the volumes rounded to two decimal places.

Type of Inscribed Solid (from largest to smallest)	**Volume (in cubic units)**
Cylinder	$V = \frac{\pi}{4}s^3 \approx 0.79s^3$
Sphere	$V = \frac{\pi}{6}s^3 \approx 0.52s^3$

Equilateral triangular prism	$V = \frac{\sqrt{3}}{4} s^3 \approx 0.43 s^3$
Square pyramid	$V = \frac{1}{3} s^3 \approx 0.33 s^3$
Cone	$V = \frac{\pi}{12} s^3 = 0.26 s^3$

By calculating these volumes all in terms of s^3, we can do a lot of comparing and analyzing of these shapes. If at the 10^{th} grade level, you feel like these equations are too complex for your students, you could use a side length of 1 unit as opposed to *s* units. A lot of interesting results can be seen in this table. For example, the cone seems to be approximately ¼ the volume of the cube and the sphere is approximately ½ the size of the cube. Also, only two shapes take up more than half the cube, the cylinder and the sphere. By inscribing objects in a basic cube, we have investigated the volume of solids at a much deeper level than your basic plug in volume calculations.

Classroom Activities

Activity #1

This activity is a way to introduce students to polyhedra. I would recommend this activity as a homework assignment or small research project as they will most likely need access to a computer. Students will gather information about what defines a platonic solid, then discover the five platonic solids, and lastly verify Euler's formula for each. Prior to this activity, students should know what defines a polyhedron and they should know and be able to apply Euler's Formula. It is important to have the five platonic solids available in your classroom for your students to touch and observe.

Platonic Solids

There are many different types of polyhedra and one of the most famous is called platonic solid. A platonic solid is a special type of polyhedra. There exist only five in the world.

1. What defines a platonic solid? Explain it in your own words!

2. What are the five different types of platonic solids? Give names and descriptions for each.

3. Since the platonic solids are examples of polyhedra, then Euler's Formula must hold true. Verify Euler's Formula for each platonic solid.

Name of Platonic Solid	Shape of face	*V*	*F*	*E*	*V + F*	*E + 2*	*Does the formula hold?*

Activity #2

In this activity, students will see how everyday objects can be modeled using mathematical solids. Students are able to relate the content of the unit to things they see in the real world. They are also able to see real world representations of surface area and volume. For example, in the case of the soda can, the surface area would represent the amount of aluminum surface needed to produce the can, and the volume would represent the amount of liquid that could fill the can. There are pictures included with each problem, but it may be helpful to also have actual examples of these objects on hand for students to see. The piece at the end, entitled EXTRA, could be used as a homework assignment or an individual project. I think it would be beneficial to hear different ideas and have them discussed as a class.

Surface Area and Volume of Everyday Figures

1. A soda can:

a. What mathematical name could you give to this object? Explain how you know.

b. What would the surface area of this figure represent? Why might that information be useful? Justify your answer.

c. What would the volume of this figure represent? Why might that information be useful?

2. A cereal box:

a. What mathematical name could you give to this object? Explain how you know.

b. What would the surface area of this figure represent? Why might that information be useful? Justify your answer.

c. What would the volume of this figure represent? Why might that information be useful? CHALLENGE: We can fill in the box with cereal, sand, or chips!

3. An ice cream cone:

a. What mathematical name could you give to this object? Explain how you know.

b. What would the surface area of this figure represent? Why might that information be useful? Justify your answer.

c. What would the volume of this figure represent? Why might that information be useful?

4. A basketball:

a. What mathematical name could you give to this object? Explain how you know.

b. What would the surface area of this figure represent? Why might that information be useful? Justify your answer.

c. What would the volume of this figure represent? Why might that information be useful?

<u>EXTRA:</u> Come up with your own examples of everyday objects that are similar to 3-D figures that we have learned about. Then answer a., b., c. from above.

Activity #3

In this activity, students will apply to surface area and volume formulas in order to find unknown measurements. The problems begin pretty straight forward but get more difficult towards the end. This activity could be given as a group project or could be broken down into warm-ups. It would be helpful to have physical representations of these objects on hand to help students visualize what they are solving for.

<u>Find the unknown piece of information</u>

For each solid, you are given certain measurements or facts. Use what you have learned to determine the answers to the unknown information. Show your work and include sketches if helpful. Assume all objects are right, meaning not oblique.

1. **Object type:** Cylinder
   Given information: radius = 7 cm, height = 4 cm
   Unknown information: *Surface Area = ?, Volume = ?*

2. **Object type:** Triangular Prism

Given information: base triangle is an equilateral triangle with side lengths of 4 inches, height of the prism = 3 inches
Unknown information: *Surface Area = ?, Volume = ?*

3. **Object type:** Square Pyramid
   Given information: Volume = 75 cubic cm, height of the pyramid = 9 cm
   Unknown information: *Area of the base = ?, side length of the base = ?, Surface Area of the pyramid = ?*

4. **Object type:** Square prism
   Given information: area of the base = 36 square inches, Total surface area of the prism = 288 square inches
   Unknown information: *height of the prism = ?, Volume = ?*

5. **Object type:** Cone
   Given information: Volume = 144*pi* cubic inches, height of the cone is twice the radius of the cone's circular base
   Unknown information: *Radius = ?, Volume = ?, Area of the base = ?*

Activity #4

In this activity, students are given a scenario with no numbers and asked to make predictions on what they think is true. For a class activity, it is a good idea to have students turn in their predictions to the teacher prior to doing the mathematics. This way, their reflection at the end will be more insightful.

Then they will investigate the answer using appropriate models and formulas. Students can use letters to represent the different dimensions of the figures, or if it will help, the teacher can put in actual numbers so they make exact calculations. After they do the math, they should take time to reflect, as a class or on their own, how accurate their predictions were. I see this being at least a full day activity or each question could be used separately as a warm-up.

<u>Make your prediction, then do the math!</u>

For each problem, you are given a geometrical scenario involving surface and/or volume and you are asked a question.

- Think about the situation and decide what you think might be true. ***Hand in your prediction to your teacher. You will revisit your prediction at the end!***
- Then investigate using pictures, numbers, calculations, etc. Find the true answer and use mathematical arguments to justify why you are right.
- At the end, reflect back on how accurate your prediction was.

1. What is the largest non-cube shape that you can fit inside a cube? You can choose from a cone, cylinder, sphere, or pyramid.

    a. Make your prediction:

    b. Do the math:

    c. How did your prediction compare to the actual answer?

2. A cylinder and a sphere have the same volume and the radius of the sphere is equal to the radius of the cylinder. What is the relationship between the height of the cylinder and the radius of the sphere?

    a. Make your prediction:

    b. Do the math:

    c. How did your prediction compare to the actual answer?

3. A ***tube*** is a cylinder with another smaller cylinder with the same axis cut out of it. A tube has a height, h, and has two radii, an inner radius, r_1, and an outer radius, r_2. An ordinary cylinder has the same outer radius of the tube, r_2, and the same height, h. If the cylinder and tube require the same amount of material to construct, what can we conclude about the inner radius, $r_{1,}$ of the tube?

    a. Make your prediction:

    b. Do the math:

    c. How did your prediction compare to the actual answer?

4. An ***hour glass*** is a cylinder that has two congruent cones whose apexes meet at the center of the cylinder. Sand is filled in the cone section of the hour glass. How

does the amount of sand in the hour glass compare to the amount of sand that could fill the entire cylinder?

a. Make your prediction:

b. Do the math:

c. How did your prediction compare to the actual answer?

5. Tennis balls are usually packaged in a cylindrical package in groups of three. If they were packaged in a rectangular prism instead, how will that package compare with the cylindrical package? You should investigate both surface area and volume.

    a. Make your prediction:

    b. Do the math:

    c. How did your prediction compare to the actual answer?

Appendix: Standards

Delaware State Standards addressed in this unit: Standard 3- Geometric Reasoning: Measurement

- Grade 9- Students will be able to demonstrate an understanding of and apply formulas for area, surface area, and volume of geometric figures including pyramids, cones, spheres, and cylinders.
- Grade 10- Students will be able to find missing dimensions of a shape given the area, volume, or surface area.
- Grade 10- Students will be able to use partitioning and formulas to find the surface area and volume of complex shapes.

NCTM Standards addressed in this unit: Geometry Standard- Use visualization, spatial reasoning, and geometric modeling to solve problems.
Grades 9-12 Expectations:

- Draw and construct representations of two- and three-dimensional geometric objects using a variety of tools

- Use geometric models to gain insights into, and answer questions in, other areas of mathematics
- Use geometric ideas to solve problems in, and gain insight into, other disciplines and other areas of interest such as art and architecture

Bibliography

Bloom's Taxonomy.

Cherico, Cindy M. "Geometry and the Design of Product Packaging." *Mathematics Teacher*, October, 2011.

Delaware Math Grade-Level Expectations, 9th through 12th grade, April 2010.

Euclid, "The Elements".

Fey, James. T., Hart, Eric W., Hirsch, Christian R., Schoen, Harold L., and Watkins, Ann E., eds. *Core-Plus Mathematics: Contemporary Mathematics in Context Course 1*. 2nd ed. New York, NY: McGraw Hill Glencoe. 2008.

NCTM, "Focus in high school mathematics: Reasoning and Sense making in Geometry," Reston, VA: National Council of Teachers of Mathematics, c2010.

NCTM Principles and Standards for teaching mathematics, 2000.

Obara, Samuel. "Where does the formula come from?" *Australian Mathematics Teacher*, 2009.

Urich, Joshua A. and Sasse, Elizabeth A. "An Appeeling Activity: Students peel oranges to explore the surface area and volume of a sphere." *Mathematics Teacher*, October, 2011.

[1] Samuel Obara, "Where does the formula come from?" Australian Mathematics Journal (2009): 33.
[2] Samuel Obara, "Where does the formula come from?" 25.

Curriculum Unit Title: Surface Area and Volume

Author: Elizabeth Denenberg

KEY LEARNING, ENDURING UNDERSTANDING, ETC.

Students will understand and apply surface area and volume formulas to complex figures.

Students will find unknown dimensions of three-dimensional objects given their surface area or volume.

Students will use geometric models to solve problems.

ESSENTIAL QUESTION(S) for the UNIT

What does calculating surface area tell you about a figure? What does calculating volume tell you about a figure?

How can you measure the surface area and volume of complex three-dimensional shapes? How can you find an unknown dimension of a three-dimensional shape?

CONCEPT A	CONCEPT B	CONCEPT C
Understand Surface are and volume	Apply surface area and volume formulas to complex solids	Find unknown dimensions of objects given surface areas and volumes
ESSENTIAL QUESTIONS A	**ESSENTIAL QUESTIONS B**	**ESSENTIAL QUESTIONS C**
What does surface area tell you about an object? What does volume tell you about an object?	How can we apply formulas to complex shapes in order to determine their surface area and volume?	How can you find an unknown dimension of an object given its surface area or volume?
VOCABULARY A	**VOCABULARY B**	**VOCABULARY C**
- Surface area, volume, units, cubic, square, lateral surface	- Cylinder, prism, cone, sphere, pyramid, right, perpendicular, parallel	- Radius, height, slant height, base,

ADDITIONAL INFORMATION/MATERIAL/TEXT/FILM/RESOURCES