Quadratic Functions as a Product of Linear Factors

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Rationale

A unit on quadratic functions tends to elicit a reaction of fear and anxiety in many students, and likely the same from just as many teachers. I think there are a number of reasons for this. First of all, the concept tends to be fairly abstract, and many units dive right into quadratic equations of the form $f(x) = ax^2 + bx + c$, and then the most foreign looking thing that many math students ever see – the quadratic formula. The formula itself leads to many opportunities for potential computational error, never mind the fact that many students have no idea what the formula represents, where it comes from, or even what they are actually using it to solve for.

Another reason for high anxiety levels with quadratics is generally the sequence of where the unit comes into play in many high school math courses. In traditional courses it is usually included in an algebra course, which might explain the intense focus on the quadratic formula, and also on factoring. The goal here is to solve for something, rather than to understand the nature of a relationship. In integrated mathematics courses, this unit typically follows a unit on linear relationships. I think that making the leap from understanding linear relationships to understanding quadratic relationships is a bigger leap than we realize.

In our school we utilize the Core Plus mathematics curriculum, which focuses on developing students as problem-solvers more than "math-doers". The content is structured around a lot of contextual situations, which does give a significant advantage when trying to teach for understanding. Unfortunately, sometimes the context does get in the way of the actual underlying mathematics, and the core level of understanding we hope the students will achieve. For example, we tend to start a quadratics unit by discussing projectile motion, and before we know it we are knee deep in concepts such as initial upward velocity, and for some reason $-16t^2$ represents gravity and we are calculating maxima and zeroes, and looking at distance over time, or is it height and distance? Or is it height and time? What happened to the constant rate of change? After stressing that linear relationships involve the relationship between 2 variables, it can get confusing trying to teach quadratics in a similar method.

So, I'm proposing that quadratics might best be understood as a resulting relationship that involves the product of two linear factors. More specifically, I propose that a deeper understanding can be gained by presenting quadratics as a product of two opposing forces. In projectile motion there is an upward force (initial upward velocity) and a downward force (gravity) and over time, the height changes depending on the effects of these two things. This is a somewhat abstract example, because curious students are forced to just accept that we are using the number $-16t^2$ as gravity's effect, without really understanding why. I think that there are many more tangible, relatable examples of how two opposing forces result in a relationship that goes up, hits a maximum, and then comes back down. For example, there is a popular problem involving an apple orchard, which is a specifically confined size, and a farmer wants to maximize his yield. But overcrowding has an effect on the yield per tree. So, for every additional tree added, then the yield per tree will decrease. These forces (the number of trees, and the number of apples per tree) are working against one another, but the goal is to find that "magic number" that will maximize the yield. Other examples of the product of opposing forces involve Profit from ticket sales (as price increases, the demand decreases), or using the area formula to find the dimensions of a picture frame or some other type of border (as the length increases, the width must decrease, or vice versa, in order to hold an area constant).

School Background and Curriculum

I teach secondary mathematics at Paul M. Hodgson Vocational-Technical High School in Newark, Delaware. Being a vocational-technical high school, students spend part of their day in their chosen vocational shop, and the remainder of the school day in core academic courses. Vocational options available to students span a broad range, including carpentry, nurse tech, plumbing, cosmetology, dental assisting, culinary, auto tech, auto body, and business tech, among others. Students and teachers at Hodgson have a distinct advantage in that the students regularly use and apply mathematical concepts in their shops. It is very unique that students are given the opportunity to make connections between their academic course content and potential applications in real world career situations. In fact, this model is something that we thrive on and have built a school culture around. Our school's motto is to "Learn It, Live It, and Apply It". Students are reminded of this motto every day during the morning announcements. For this reason, we, as teachers must always continue to look for direct connections between our course content and vocational-technical applications in order to make things relevant for our students.

The unit on quadratic functions will be taught over the course of approximately 30 school days. At Hodgson we use the Core Plus Mathematics Curriculum as our primary series of textbooks. This semester we are facing a somewhat unique situation. Over the past few years, we have utilized an Integrated Math I, II, III, and IV course structure in a way that worked fairly well with our block scheduling system. In order to make the curriculum work with our schedule, and the fact that our courses are built around two semesters, we had to choose certain units. So, our Integrated Math I course did not necessarily alight perfectly with the entire Core Plus Course I textbook. Each of our Integrated Math courses only covered portions of the textbook, so that by the end of

Integrated Math IV, students have basically completed Core Plus book three. As a district, we recently decided that we needed to realign our courses in order to ensure that the material we cover appropriately prepares students according to the Common Core State Standards. So, this has presented a unique challenge for students and teachers this semester.

Quadratics Unit in Semesters Past

The quadratics unit that we teach at Hodgson is included in both the first and the second books in the Core Plus series. Historically, these two quadratics units have been covered at two separate points in a student's high school career. The unit from book one has been included in our Integrated Math II course, and then when students are in Integrated Math III they cover the next unit on quadratics, which is in book two of the Core Plus series. There are pros and cons to having the topic broken out into two separate parts like this. The major advantage has been reinforcement. Having a unit on quadratics once during freshmen year and then building on this during sophomore year gives students more longitudinal exposure to the content. They get the chance to re-visit the content rather than learning it once and then perhaps forgetting it.

Lesson Objectives	On Your Own Assignments*	Suggested Pacing	Materials
Lesson 1 ***Quadratic Patterns*** • Determine patterns of change associated with quadratic functions • Use tables of values and graphs to estimate answers for questions about situations modeled by quadratic functions • Describe the effects of each parameter in the function rule $y = ax^2 + bx + c$	**After Investigation 1:** Choose two of A1–A4, C9, R15, E19 or E20, Rv27–Rv29 **After Investigation 2:** A5, A6, C10, R16, Rv30–Rv33 **After Investigation 3:** A7, A8, choose two of C11–C14, R17, R18, choose one of E21–E26, Rv34, Rv35	8 days (including quizzing)	• Spreadsheet software available in *CPMP-Tools* • Unit Resources
Lesson 2 ***Equivalent Quadratic Expressions*** • Find symbolic rules for quadratic functions using data modeling and reasoning • Determine whether two given quadratic expressions are equivalent • Decide on most useful equivalent forms of quadratics for different question types • Create equivalent quadratic expressions by expanding products of linear factors • Factor quadratic expressions by extracting common linear factors	**After Investigation 1:** A1–A3, C9, C10, R14, E18, Rv26–Rv28 **After Investigation 2:** A4–A8, C11–C13, R15–R17, choice of E19–E25, Rv29, Rv30	5 days (including quizzes)	• CAS technology available in *CPMP-Tools* • Unit Resources
Lesson 3 ***Solving Quadratic Equations*** • Write quadratic equations and inequalities to express questions about quadratic functions • Find exact values of solutions for quadratic equations in the form $ax^2 + c = d$ and $ax^2 + bx = 0$ by reasoning and factoring • Relate factored forms of quadratic expressions to x-intercepts of graphs for the related functions • Solve quadratic equations by using the quadratic formula • Describe the possible number of real solutions for quadratic equations and illustrate the possibilities with graphs	**After Investigation 1:** A1, A2 or A3, A4–A6, C9–C11, R15, choose one of E19–E21, Rv27–Rv30 **After Investigation 2:** A7, A8, C12–C14, R16–R18, choose one of E22–E26, Rv31–Rv34	6 days (including quizzes)	• CAS technology for E24 • Spreadsheet software for E25 • Unit Resources

Table 1[1]

Core Plus Course One

The first exposure students have to quadratics in this format is in the context of projectile motion. The Core Plus curriculum jumps right into the content by investigating the annual "Punkin' Chunkin'" festival in lower Delaware. Students explore patterns of projectiles that begin by just falling, and then projectiles that have an initial upward velocity. They are introduced to the form $y=ax^2+bx+c$ and then they explore and describe the effects of changing the parameters a, b, and c. Students are given a very brief summary of Galileo's experiments and told that "gravity exerts a force on any free-falling object so that d, the distance fallen, will be related to time t, by the function

$d=16t^2$ (time in seconds and distance in feet)."

They are then told that, "The model ignores the resisting effects of the air as the pumpkin falls. But, for fairly compact and heavy objects, the function $d=16t^2$ describes motion of falling bodies quite well."[2]

Using this information, students explore patterns in tables and graphs to develop a basic understanding of projectile motion using Galileo's discovery of the distance-time relationship. They use tables and graphs to solve equations relating distance (or height) and time for falling objects. They then go on to study situations, such as suspension bridges, where the parabola opens facing upwards, and are led to discover that sign of the "a" term is what leads to the direction of a parabola. In this lesson there is also a crucial example which I think tends to unfortunately be glossed over much of the time – an example involving profit as a function of ticket price. This problem set is structured in such a way where students are led to the idea that they need to multiply the number of tickets sold by the ticket price in order to calculate profit. But the challenge is that the number of tickets sold is a linear function, which itself depends on the ticket price. As you will read further on in this discussion, I think this example is one that could really be elaborated on and in fact an entire unit I think could be built on a premise such as this one.

In the next lesson, students are introduced to equivalent quadratic expressions through the utilization of the distributive property. They first explore equivalent expressions using tables and graphs to determine if expressions are equivalent or not. They eventually will formalize the algebraic method using the distributive property. Here is where I think the major opportunity lies. There is an investigation of the income and expenses that go into putting on a high school dance. This example represents the students' first exposure to a quadratic expression as the product of linear factors. Further on in this paper, I will propose that this makes an ideal starting point for a unit on quadratics since it relates directly back to work students have done on linear functions. In fact, there is also another important connection in this unit, where students are comparing what happens when you add two linear functions to what happens when you multiply linear functions. This is a really powerful connection to make and I think it is kind of a shame that it is utilized in a fairly narrow way – that is just to teach the distributive property and equivalent expressions. To me, this represents one of the most important utilizations of quadratics in every day life. But I digress. Anyway, the remainder of this lesson is instruction and practice on how to use the distributive property with an "x^2" term, and then on doing "double distribution". This algebraic practice is definitely necessary, and will serve to be doubly important for students to be able to factor quadratic expressions later on.

The third lesson of the Book One quadratics unit is when students are officially introduced to solving quadratics functions. With that said, they initially are limited to solving quadratics that are missing either the "b" term or the "c" term – that is to say, they are solving the general forms $ax^2 + c = d$ and $ax^2 + bx = 0$. Also, during this section students use the symmetry of a parabola to find maxima and minima. There is a very brief explanation of the zero product property:[3]

Check the reasoning in this proposed solution of the equation.

i. The expression $24t - 16t^2$ is equivalent to $8t(3 - 2t)$. Why?

ii. The expression $8t(3 - 2t)$ will equal 0 when $t = 0$ and when $3 - 2t = 0$. Why?

iii. So, the solutions of the equation $24t - 16t^2 = 0$ will be 0 and 1.5. Why?

Again, this is another major revelation toward building an understanding of how to solve quadratic functions in factored form. The screen shot above is from the section that introduces how to solve quadratics of the form $ax^2+bx = 0$. I really think there is an opportunity here to go into this concept in a more robust way so that students really have a deeper understanding of why setting each of the factors equal to 0 is useful for solving. This topic is of major importance later, so I think it would make a lot of sense to put more focus on it at this point.

Finally, the last part of the Book One quadratics unit is on how to use the quadratic formula to solve quadratics of the form $ax^2 + bx + c$. The introduction is very basic and focuses mainly on application of the formula. The text foreshadows the fact that students will prove where the formula comes from, and then describes how to use the formula very procedurally, from here, students have the opportunity to practice using the quadratic formula to solve for the zeroes. It also briefly gets into the idea of using the formula to determine if there are zero, one, or two solutions.

Many problems that require solving quadratic equations involve *trinomial* expressions like $15 + 90t - 16t^2$ that are not easily expressed in equivalent factored forms. So, solving equations like

$$15 + 90t - 16t^2 = 100$$

(When is a flying pumpkin 100 feet above the ground?)

is not as easy as solving equations like those in Investigation 1.

Fortunately, there is a **quadratic formula** that shows how to find all solutions of any quadratic equation in the form $ax^2 + bx + c = 0$. For any such equation, the solutions are

$$x = \frac{-b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a} \text{ and } x = \frac{-b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}.$$

In Course 3 of *Core-Plus Mathematics*, you will prove that the quadratic formula gives the solutions to any quadratic equation. For now, to use the quadratic formula in any particular case, all you have to do is

- be sure that the quadratic expression is set equal to 0 as is prescribed by the formula;
- identify the values of a, b, and c; and
- substitute those values where they occur in the formula.

So, at this point, students should have a working understanding of:

- Writing and solving projectile motion problems involving quadratic equations with $-16t^2$ as the effect of gravity
- Re-writing quadratic expressions from factored form to equivalent standard form
- Solving $ax^2+bx=0$ and $ax^2+c=d$ problems algebraically using symmetry and the zero product property
- Solving $ax^2+bx+c=d$ problems using the quadratic formula
- Using the quadratic formula to determine how many solutions a quadratic function has
- Describing the effects of the parameters a, b, and c on a quadratic graph

What students have NOT covered at this point:

- Re-writing a quadratic expression from standard form into equivalent factored form
- Using factored form to solve for the zeroes of a quadratic function
- Explaining the differences and advantages to factored form vs. standard form
- Using factoring to locate the vertex, and explaining where the minimum or maximum is
- Using zeroes to write a curve-fitting function in factored form
- Factoring where the “a” term is not 1
- Solving nonlinear systems of equations
- Robust application problems involving the product of linear factors
- Solving quadratics by completing the square

Core Plus Course Two

Some of the bullet points of topics that have not been covered above are addressed in the second textbook in the Core Plus series.

Lesson Objectives	On Your Own Assignments*	Suggested Pacing	Materials
Lesson 1 ***Quadratic Functions, Expressions, and Equations*** • Distinguish relationships between variables that are functions from those that are not • Use $f(x)$ notation to represent functions and the common questions about functions that arise in applied problems • Identify domain and range of functions • Construct rules for quadratic functions based on given properties such as x-intercepts, y-intercept, and maximum/minimum point • Write quadratic expressions in equivalent expanded or factored form • Solve quadratic equations by factoring, by applying the quadratic formula, or by a CAS	**After Investigation 1:** A1–A3, A4 or A5, A6, C16, C17, choose one of R25–R27, E34 or E35 **After Investigation 2:** A7–A10, C17, R28, R29, Rv43–Rv47 **After Investigation 3:** A11, A12, choose three of C18–C23, R30, E36 or E37, E38, Rv48–Rv50 **After Investigation 4:** A13–A15, C24, choose two of R31–R33, choose one of E39–E42, Rv51, Rv52	10 days (including assessment)	• Computer software or calculators with CAS capatability such as the ***CPMP-Tools*** algebra software • Unit Resources
Lesson 2 ***Nonlinear Systems of Equations*** • Write an equation or inequality to represent a question about a "real-life" situation involving a comparison between a linear function and either an inverse variation or quadratic function • Estimate solutions to equations in the form $ax + b = \frac{k}{x}$ using tables or graphs and solve algebraically • Estimate solutions to equations in the form $mx + d = ax^2 + bx + c$ using tables or graphs and solve algebraically	**After Investigation 1:** A1–A4, R15, R16, E19, Rv25–Rv28 **After Investigation 2:** A5 or A6, A7, A8, choose two of C11–C14, R17, R18, choose two of E20–E24, Rv29	6 days (including assessment)	• Computer software or calculators with CAS capatability such as the ***CPMP-Tools*** algebra software • Unit Resources

The primary distinction between Course 1 and Course 2 is that course 2 dives more heavily into factoring quadratics of the form ax^2+bx+c[4]. It also revisits the distributive property and expanding, so students should be very fluent at manipulating back and forth between factored form and standard form for any quadratic function. The other primary distinction that course two goes into is constructing rules for quadratic functions based on given features of a graph. For example, when given two x-intercepts and a vertex, students should be able to work backwards to write the linear factors and then multiply by a constant (scale factor) to adjust the "height" of a graph. As mentioned, there is a distinct advantage to revisiting the content in the next semester. But with that said, there are some drawbacks. One primary drawback includes the fact that re-teaching concepts such as distributing and expanding can use up instructional time that might best be used elsewhere. Another drawback is resistance to new material. For example, I've had students who get very attached to the quadratic formula and so comfortable using it that they see no need to learn how to factor, and thus always solve for the zeroes using the quadratic formula. So, rather than being seen as a unit that builds on the previous course's material, it can be seen as a unit that is redundant, or unattached. Typically, in the Integrated Math 3 course where we teach this unit, it is surrounded by a unit on coordinate geometry and a unit on trigonometry. So having a quadratic unit, which is heavily focused on algebraic reasoning, sandwiched between two units that utilize more geometric concepts, does not necessarily flow as smoothly as it could.

The quadratics unit in Course 2 starts out by having students break down a quadratic graph into its key parts – so they are identifying the y-intercept, x-intercept(s), line of symmetry and vertex (max or min). They will then use this information to construct quadratic functions in factored form. The begin by writing functions in factored form f(x)=(x-m)(x-n), but then they have to realize how to adjust this rule to match different parabolas with the same intercepts by finding the constant in the form f(x)=a(x-m)(x-n). Also during this lesson is where students make the connection that the vertex / line of symmetry occurs halfway between the two x-intercepts. So, the main theme of this lesson is to really drive home the relationship between the factored form of a quadratic and how this relates to key features of the graph. The underlying theme here should therefore be on highlighting these features as advantages of the factored form.

From here, the textbook officially makes its first foray into factoring quadratics of the form ax^2+bx+c, after a quick refresher on expanding using distribution. The focus here begins simply with the skill of factoring with plenty of procedural practice problems. There are also small allusions made to some special examples such as $(x+a)^2$ and difference of squares problems such as $x^2 – 9$ being factored as (x+3)(x-3). During this lesson I think there is another important missed connection. As mentioned, this topic presents factoring in a procedural way, giving plenty of opportunity to practice. Thrown into these practice problems, are examples of the form $ax^2+bx=0$ and $ax^2+c=0$, but there is not really a connection made to the fact that students already learned these in the previous course. I think there is an opportunity to present factoring in a way that is built on something they already know, rather than as an alternative method for solving. I see this happen frequently, where students get so focused on the idea of factoring that they forget how to solve these other two types of problems, when really they are actually easier. So, this is probably the biggest downside to separating the quadratics content over two different courses. Connections like this are missed, and then students get so good at solving one type of problem, but are unable to distinguish between different solution methods.

After a few days practicing how to re-write quadratics from standard form into factored form, the connection is finally made for how to use factoring to solve an equation. As previously mentioned, many students are resistant to this because they are comfortable using the quadratic formula and they do not see factoring as any easier. Factoring requires critical thinking (i.e. searching for the factors of "c" that add up to "b", can sometimes be frustrating, and actually might not always even exist), whereas the quadratic formula is merely a matter of plugging in numbers. This is the primary reason that I will propose that factoring be covered right at the beginning of a quadratics unit. By the time we get to factoring, students have become set in their ways, or they are mentally exhausted of quadratics and ready to move on to something new!

Quadratics for the coming school year

As previously mentioned, my school district went through the process of realigning our courses this year to better align with the Common Core Standards, and to hopefully better prepare students for Smarter Balance testing, which should begin next school year. This means changes for every teacher, and rethinking the way we go about things. It also means students are coming from one class to the next prepared differently than in semesters past. So, with this realignment, there has been some adjustment to the way that quadratics is being covered. And this semester, being a transition year, is creating some even more unique challenges.

So, moving forward, our realigned course guides state that in Integrated Math 1, students will be exposed to just the first lesson from the Core Plus Course 1 textbook. So, coming into Integrated Math 2 they will have only a very basic understanding of the structure of a quadratic equation and the key elements of a parabola. They will not have been exposed to expanding or factoring, and they will not have been exposed to solving algebraically in any way. They will just be familiar with how to read tables and graphs to answer questions. So, this leaves all of the rest for Integrated Math 2. This is a larger stress on the IM-2 teacher, but perhaps presents many advantages in the form of more cohesiveness and the ability to go deeper on certain concepts. It also allows for more flexibility in the sequencing of the content.

Activities

Before we launch into the actual investigation, I will begin with a warm-up intended to help students discover a quick and easy way to solve problems involving a linear equation set equal to zero:

Solve for x. See if you notice a pattern.

$x+3 = 0$	$2x + 4 = 0$
$3x - 2 = 0$	$3x + 5 = 0$

$ax + b = 0$

This will prime the students for the type of solving they will need to do later when finding x-intercepts from the factored form of a quadratic. They will discover that x=-b/a and the hope is that this will be a useful tool moving forward. More importantly it will refresh their memory on linear equations and algebraic reasoning. These also serve the purpose of being fairly easy problems to warm-up with, thus building students' confidence going into the launch.

Launch / Activity #1

Students will investigate the following problem:

> An orchard contains 30 apple trees, each of which yields approximately 400 apples over the growing season. The owner plans to add more trees, but the experts advise that because of crowding, each new tree will reduce the average yield per tree by about 10 apples over the growing season. How many trees should be added to maximize the total yield of apples, and what is the maximum yield?

This problem will be posed as is, with very little scaffolding. But there will be opportunity for the students to ask clarifying questions. The need for clarification could be prevented by asking students 2 questions: First, how many total apples will there be currently? (30x400=12000 apples) Secondly, what will the yield per tree be if there are 31 trees? (400-10(1)=390). These two prompting questions should help to clarify the fact that multiplication is required between the two key items, and also that as one variable increases, the other decreases. This activity will work best using a Think-Pair-Share format, where they are given about 5 minutes to work silently and individually, then another 5 minutes to work with a partner, comparing their thought process, work, and answers. After this, the teacher should facilitate a share-out, selecting and sequencing anticipated responses.

Anticipated methods might include trial and error, using tables, or writing a formula. As we work through the lesson, the end goal should be for students to discover that an expression can be written to describe the situation in the form (30+x)(400-10x), where x represents the number of additional trees. The most common approach will likely be to use a table. Generally students will quickly try multiplying 30*400, and then try multiplying 31*390, and after doing one or two iterations the pattern will become apparent to them. This problem is nice because a satisfactory answer can be found using recursion after only 5 iterations. So it sets the level of potential frustration pretty low, but also lays a foundation for lots of follow-up conversation on how to generalize.

The share-out should be focused around the idea of using the recursive pattern to find that the 5^{th} additional tree will give a total of 12,250 total apples. Ideally, at least one pair of students will have shown their work in some type of tabular format, similar to the one below. Clearly, any share-out should be tailored to explore the ideas that the students in the class have come up with. But at some point, there should be a discussion of how tables can be used to come up with a solution (similar to the table below, which expresses the relationship between # of trees, and # of apples per tree). When expressing this pattern in table form, it may help to make it easier to see that each column represents a linear equation.

After sharing out ideas, and making sure that a table is discussed, then the next probing questions will dive deeper, and help expand on the larger overall pattern, rather than confining only to the first 5 iterations.

Questions to think about...

1 - What equation could model this situation?

2 - How many additional trees until the total yield will be zero apples?

3 - Is there another possible situation where there would be zero apples (hint: it's possible to remove trees)

4 - How is the maximum point related to the zero(s)?

Posing these questions as a follow-up is the crucial element of this lesson. The main goal is to help students understand that a quadratic can be created as the product of two linear factors. At the same time, it will also illuminate the pattern that when you multiply one increasing linear function, and a decreasing linear function, the result will be a parabola that increases and then decreases. When given in context, this should help students realize that there are two different ways to end up getting zero total apples. First of all, they can continue adding trees until the effect of over-crowding is so much that each tree grows no apples, resulting in zero apples. The other option is to simply have no trees (which could lead to an interesting conversation about theory vs practicality), thus ending up with zero total apples. The most important connection is that these two zero points correspond to the zero

trees added	# of trees	# of apples per tree	total # of apples
0	30	400	12000
1	31	390	12090
2	32	380	12160
3	33	370	12210
4	34	360	12240
5	35	350	12250
6	36	340	12240
7	37	330	12210
8	38	320	12160
9	39	310	12090
10	40	300	12000
x	30+x	400-10x	(30+x)(400-10x)

Going back to the table format is a helpful way to work through the process of coming up with the factored form of the equation. Understanding that the “total # of apples” column is derived from multiplying the “# of trees” and “# of apples per tree” columns, along with the fact that each of those columns can be expressed as a linear equation should illuminate the idea that factored form is the product of two linear equations. Note that exploring this topic should take time, and students should be given the opportunity to develop a deep understanding through independent work and collaborating with a partner. So often we tend to just teach that the vertex is halfway in between the x-intercepts, and when done this way I’ve had issues with students really understanding and retaining this. But when given the opportunity to explore and think about this, it turns out to be pretty intuitive that the vertex should occur halfway between the zeroes. This is a powerful tool to be used later when we get into factoring and solving, but for now it’s important to build the understanding of what happens when you multiply two linear equations. If necessary, the extension questions should be something that can be finished for homework, rather than rushing through the share-out and moving on. Another potential follow-up, or perhaps a summary/”ticket out the door” would be to ask students to sketch what they think a graph of the situation might look like.

At this point, we are laying the foundation for the bulk of what will be covered in the unit – factored form, identifying x-intercepts, vertex, and the overall pattern of a quadratic. The follow up to this activity will be for students to practice using the zero product property, and understanding why and how this is useful. But first, it is helpful to reinforce the relationships developed from the apple orchard problem by working through another example. So, the next activity will work in a similar fashion. After having worked through the previous problem, more students should find success in the follow-up problem. Initially, I structured this the same as the orchard problem – i.e. first have the students find the maximum, then as an extension, find the zeroes. But, at teacher discretion, the students may be able to explore all the questions at once and then share out the results from the entire problem.

Activity #2 / The school dance:

Tickets to a school dance cost $4 and the projected attendance is 300 people. For every $0.10 increase in ticket price, the dance committee projects that attendance will decrease by 5 people.

a) What ticket price will maximize revenue (income)?

b) Determine the greatest possible revenue?

c) What are the two situations that will end up giving a revenue of zero dollars?

d) How is the maximum related to these "zeroes"?

e) What equation could model this situation?

Being the second problem of a similar nature, this should move a little quicker, but at the same time, there is an opportunity to go a little deeper. This could be the opportunity to actually have students create a graph, either by hand, or using the calculator, and to identify key points. If this is done, then effort should be made to connect these key points to the table. It is also a time to spend reinforcing the fact that the vertex is halfway between the two x-intercepts. In addition, the y-intercept could be introduced at this point, or at least alluded to, and then revisited later. After working through this problem, then the next step will be to really drive home the idea of the zero product property. Hopefully after working through two richly contextual problems where there is a firm understanding of two ways to "get zero" as a solution, then students will have a better time later on when we solve problems by factoring. Again, the rationale behind the level of depth here is that so many times I've seen students not really understand why you can just "break the factors apart and set them equal to zero". Or worse yet, some students just memorize that the solutions are the opposite of the numbers in factored form. Hopefully the apple orchard problem and the high school dance problem clears up the confusion on why there might be two "zeroes" and how factored form can be used to find them.

So, the next step is to formalize the Zero Product Property:

Zero Product Property.

When you multiply two numbers and get an answer of zero, what does this mean about the two numbers?

Again, here is another chance to really have the students think about the definition, rather than memorizing a mathematical property. There is also an opportunity here to clarify why this property only works when the product is zero and not for any other number. For example, if two integers are multiplied to equal 12, we don't know whether those two numbers are 3 and 4, or 12 and 1, or 6 and 2, or non-integers. But, when two numbers multiply to equal 0, we know that one of those numbers MUST BE ZERO.

So, $(x+30)(400-10x) = 0$ means one of the **factors** must be 0.

So, the product $(x+30)(400-10x)=0$

when either...

$x+30 = 0$ or $400-10x = 0$

$x = {}^{-}30$ $\quad\quad$ $x = 40$

From here, the next step is to provide some practice. The following worksheet is ideal.

Activity #3 / Zero Product Property

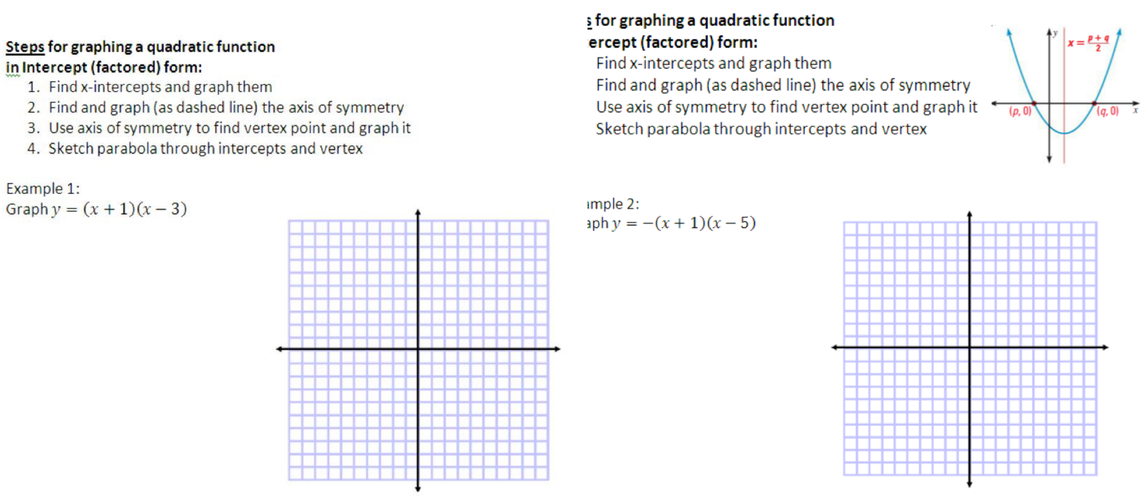
Zero Product Property Worksheet

Solve

1. $(x-2)(x-3)=0$ __________
2. $(x+5)(x+3)=0$ __________
3. $x(x-1)=0$ __________
4. $(2x-2)(3x+3)=0$ __________
5. $2x(3x-6)=0$ __________
6. $(-x+5)(x-5)=0$ __________
7. $x(x+8)=0$ __________
8. $x(2x-1)(3x+9)=0$ __________
9. $(2x-1)(3x+1)(x+2)=0$ __________
10. $(1/2\ x+4)(1/3x-3)x=0$ __________

Activity #4 / Connection to Graphing

The primary goal of the next day, after students understand the zero product property, and the general relationship involving the product of linear factors, will be to build an understanding of the key parts of a quadratic graph. A quick warm-up should refresh the zero product property, and then go over the previous worksheet as a class. Begin the lesson with a toolkit, defining the following terms: quadratic function, standard form, factored form, parabola, x-intercept(s), axis of symmetry, vertex. Here is where the formula for axis of symmetry can be generalized as the mean of the x-intercepts. It should also be reinforced that the vertex lies on this line and can be found by plugging the x-coordinate into the original equation. After this, launch into examples on how to create graphs from only the information given in factored form. Chunking works well here – give students 5 minutes to either work silently or with a partner on each of the following problems.


Steps for graphing a quadratic function in Intercept (factored) form:

1. Find x-intercepts and graph them
2. Find and graph (as dashed line) the axis of symmetry
3. Use axis of symmetry to find vertex point and graph it
4. Sketch parabola through intercepts and vertex

Example 1:
Graph $y = (x+1)(x-3)$

s for graphing a quadratic function
ercept (factored) form:

Find x-intercepts and graph them
Find and graph (as dashed line) the axis of symmetry
Use axis of symmetry to find vertex point and graph it
Sketch parabola through intercepts and vertex

mple 2:
aph $y = -(x+1)(x-5)$

Key areas to stress during these exercises are how to tell whether the graph "opens" up, or down (the negative coefficient in front), and how to find the y-intercept. The y-intercept is the connection and transition to the next key learning. Focus on how the y-intercept can be found by substituting a zero in for both "x's". But the other way to identify the y-intercept is that it is the "c" term in standard form. So, this is the perfect opportunity to transition to how to rewrite quadratic equations from factored form to standard form. Since we need to use standard form to easily find the y-intercept, then we should learn how to rewrite factored form equations into standard form. This is fairly procedural, but teaching in multiple ways may benefit different learning styles. So, the following two worksheets will be beneficial – the first one is a "toolkit" that can be kept for reference, and the second is a practice worksheet.

Multiplying Binomials Toolkit

(x + 2)(x + 3)

Double Distributive Method

$$
\begin{aligned}
(x + 2)(x + 3) &= x(x + 3) + 2(x + 3) \\
&= x^2 + 3x + 2x + 6 \\
&= x^2 + 5x + 6
\end{aligned}
$$

- Split the first binomial up into two terms
- Multiply the first term by the second binomial
- Multiply the second term by the second binomial

Punnett Square Method

	x	$+3$
x	x^2	$3x$
$+2$	$2x$	6

$x^2 + 3x + 2x + 6$

Answer: $x^2 + 5x + 6$

- Split each binomial into two terms
- Put the terms on the top and left-hand side of the "window"
- Multiply just as you would a normal multiplication chart
- Put all the terms back together and combine "like terms"

F.O.I.L. Method

First:	$x \cdot x$	$= x^2$
Outer:	$x \cdot {}^{+}2$	$= 2x$
Inner:	${}^{+}3 \cdot x$	$= 3x$
Last:	${}^{+}3 \cdot {}^{+}2$	$= 6$

$x^2 + 2x + 3x + 6$

Answer: $x^2 + 5x + 6$

- First: multiply the first term in each binomial
  (**x** + 3)(**x** + 2)
- Outer: multiply the outside terms in each binomial
  (**x** + 3)(x **+ 2**)
- Inner: multiply the inside terms in each binomial
  (x **+ 3**)(**x** + 2)
- Last: multiply the last term in each binomial
  (x **+ 3**)(x **+ 2**)
- Put all the terms back together and combine "like terms"

Activity #5 / Multiplying Binomials / Rewriting in Standard Form

IM III Multiplying Binomials Practice Worksheet

1) $(x + 3)(x + 6)$

2) $(x - 4)(x + 6)$

3) $(x - 4)(x - 1)$

4) $(2x + 7)(x - 3)$

5) $(2x - 10)(x - 10)$

6) $(2x + 8)(2x - 8)$

7) $(x - 4)(x - 4)$

8) $(3x + 1)(x - 6)$

9) $(x + y)(2x + y)$

10) $(2x - y)(x - y)$

This is all setting the students up for being able to factor a standard form trinomial. Laying the foundation of factored form first should really help the students have a better understanding of the advantages of the two forms, and now factoring can be presented as just the opposite of what they've already done, or "undoing" the double distribution. It is always useful to stop and summarize with the students. At this point it should be made clear that they are able to use factored form to understand key pieces of a graph. They can use standard form to understand key features of a graph. And they can rewrite equations from factored form to standard form. The one missing piece is to be able to write equations from standard form to factored form.

Activity #6 / Factoring

Unit 5, Lesson 1, Inv. 3 – Intro to Factoring Trinomials

FACTORING IS THE REVERSE of multiplying. Being skillful in factoring, then, depends upon your skill of multiplying. You have just finished reviewing how to EXPAND, or multiply, two binomials by the:

- Double distributive method
- F.O.I.L. method
- Punnett Square method

Consider this TRINOMIAL (another name for a quadratic expression): $x^2 + 4x - 5$
It can be factored as the product of two binomials: ⟶ $(? \pm ?)(? \pm ?)$

The first term of each binomial will be the FACTORS of "x^2". How can we produce x^2? There is only one way: $x \cdot x$. So, place "x" in the first TERM of each binomial: $(x \pm ?)(x \pm ?)$

The last term of each binomial will be the FACTORS of -5. What are the factors of -5?

+1 and -5 ⟶ plugging them in, this would give us… $(x + 1)(x - 5)$
-1 and +5 ⟶ plugging them in, this would give us… $(x - 1)(x + 5)$

BUT we CAN'T have TWO ANSWERS that BOTH lead to $x^2 + 4x - 5$! HENCE the need to understand multiplying!

So, which product gives us the desired answer? It really boils down to the two "x" terms when multiplying:

$(x + 1)(x - 5) = x^2 - 5x + 1x - 5$ Do you notice the two "x" terms that can be combined? They give us $x^2 - 4x - 5$. **THAT'S NOT RIGHT!**

$(x - 1)(x + 5) = x^2 + 5x - 1x - 5$ Do you notice the two "x" terms that can be combined? That gives us $x^2 + 4x - 5$. **CORRECT!**

Therefore, the correct factored form for $x^2 + 4x - 5$ is $(x - 1)(x + 5)$. The order of the factors does not matter. I could have easily written $(x + 5)(x - 1)$ and still been correct.

IN SUMMARY, skill in factoring depends on skill in multiplying – particularly in picking out the MIDDLE TERM!

Let's begin by practicing some skills needed for factoring.

Practice Set #1

Place the correct SIGN in each factor that will produce the given middle term.

a) $x^2 + 5x + 6 =$ $(x \quad 2)(x \quad 3)$

b) $x^2 - x - 6 =$ $(x \quad 3)(x \quad 2)$

c) $x^2 + x - 6 =$ $(x \quad 3)(x \quad 2)$

d) $x^2 - 5x + 6 =$ $(x \quad 3)(x \quad 2)$

e) $x^2 + 7x + 6 =$ $(x \quad 1)(x \quad 6)$

f) $x^2 - 7x + 6 =$ $(x \quad 1)(x \quad 6)$

g) $x^2 + 5x - 6 =$ $(x \quad 1)(x \quad 6)$

h) $x^2 - 5x - 6 =$ $(x \quad 1)(x \quad 6)$

Practice Set #2

Give all the factors of the number in the first box. Then choose which set of factors gives the SUM of the number in the second box. The first one is done for you.

Give ALL the FACTORS of ...		Pick which set of factors will give the SUM of....	
+9	***+1 & +9 -1 & -9 +3 & +3 -3 & -3***	-10	***-1 & -9***
-2		-1	
+10		-11	
-14		-5	
-12		+4	
+20		-12	

Give ALL the FACTORS of ...		Pick which set of factors will give the SUM of....	
+18		+9	
+6		-5	
-16		-6	
+18		+11	
-12		+1	
-24		+2	

	Copy the problem $Ax^2 + Bx + C$	List ALL the factors of **C**	Pick the set of factors that give the SUM of **B**	Write the factored form	Did you check your answer? (Multiply the two binomials)
Ex. 1	$x^2 - 6x + 8$	+1,+8 +4, +2 -1, -8 -4, -2	-4 and -2	$(x - 4)(x - 2)$	✓
Ex. 2	$a^2 + 3a - 10$	+1,-10 -1, +10 +2, -5 -2, +5			
Prac. 1	$d^2 + 5d + 6$				
Prac. 2	$p^2 - 8p + 12$				
Prac. 3	$y^2 - 4p - 45$				
Prac. 4	$n^2 + 5n - 24$				

You will notice some patterns when factoring trinomials:

$Ax^2 + Bx + C$ In this case, both **B** and **C** are positive, so your factors will be positive, too.
Answers will look like: $(x + \mathbf{m})(x + \mathbf{n})$

$Ax^2 - Bx + C$ In this case, **C** is positive but **B** is negative, so your factors will both be negative (remember: a neg * neg = pos!).
Answers will look like: $(x - \mathbf{m})(x - \mathbf{n})$

$Ax^2 + Bx - C$ In this case, **C** is negative, so one of your factors must be negative as well. Since **B** is positive, the ***larger factor will be positive***.
Answers will look like: $(x + \mathbf{m})(x - \mathbf{n})$

$Ax^2 - Bx - C$ In this case, **C** is negative, so one of your factors must be negative as well. Since **B** is negative, the ***larger factor will be negative***.
Answers will also look like: $(x + \mathbf{m})(x - \mathbf{n})$

Complete the following problems from your book using this method:
p338, #9

	Copy the problem $Ax^2 + Bx + C$	List ALL the factors of **C**	Pick the set of factors that give the SUM of **B**	Write the factored form	Did you check your answer? (Multiply the two binomials)
a					
b					
c					
d					
e					
f					
g					
h					
i					
j					

Now do **p339, #10, #11, and #12** in the same way (on loose leaf).

Note: #10h, #12g, and #12h are different types of problems:

They are binomials and will only factor to: ? (? ± ?)

Appendix: Common Core State Standards

While a unit as broad as Quadratic Functions can touch on a variety of standards from algebra, functions, and even geometry, included here is a non-exhaustive list of the primary standards that this unit should address.

Interpret the structure of expressions.

- CCSS.Math.Content.HSA-SSE.A.1
- CCSS.Math.Content.HSA-SSE.A.1a
- CCSS.Math.Content.HSA-SSE.A.1b
- CCSS.Math.Content.HSA-SSE.A.2

Write expressions in equivalent forms to solve problems.

- CCSS.Math.Content.HSA-SSE.B.3
- CCSS.Math.Content.HSA-SSE.B.3a
- CCSS.Math.Content.HSA-SSE.B.3b

Perform arithmetic operations on polynomials.

- CCSS.Math.Content.HSA-APR.A.1

Understand the relationship between zeros and factors of polynomials.

- CCSS.Math.Content.HSA-APR.B.2
- CCSS.Math.Content.HSA-APR.B.3

Use polynomial identities to solve problems.

- CCSS.Math.Content.HSA-APR.C.4
- CCSS.Math.Content.HSA-APR.C.5

Create equations that describe numbers or relationships.

- CCSS.Math.Content.HSA-CED.A.1
- CCSS.Math.Content.HSA-CED.A.2
- CCSS.Math.Content.HSA-CED.A.3
- CCSS.Math.Content.HSA-CED.A.4

Solve equations and inequalities in one variable.

- CCSS.Math.Content.HSA-REI.B.3
- CCSS.Math.Content.HSA-REI.B.4
- CCSS.Math.Content.HSA-REI.B.4a
- CCSS.Math.Content.HSA-REI.B.4b

Solve systems of equations.

- CCSS.Math.Content.HSA-REI.C.6

Represent and solve equations and inequalities graphically.

- CCSS.Math.Content.HSA-REI.D.10
- CCSS.Math.Content.HSA-REI.D.11

Analyze functions using different representations.

CCSS.Math.Content.HSF-IF.C.7
CCSS.Math.Content.HSF-IF.C.7a
CCSS.Math.Content.HSF-IF.C.8
CCSS.Math.Content.HSF-IF.C.8a

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[2] Hirsch, *Core Plus mathematics,* Book One, 464
[3] Hirsch, *Core Plus mathematics,* Book One, 512
[4] Hirsch, *Core Plus mathematics,* Book Two, T325B

Curriculum Unit Title: Quadratic Functions as a Product of Linear Factors

Author: Michael A. Malaney

KEY LEARNING, ENDURING UNDERSTANDING, ETC.

- Students will understand what pattern is created when two linear equations are multiplied by each other
- Students will understand how to write quadratic equations in factored form and standard form, and how to use these to create graphs
- Students will understand how to use factoring to rewrite and solve quadratic equations

ESSENTIAL QUESTION(S) for the UNIT

What is a quadratic function, and how does it relate to linear functions?

What do graphs of quadratic functions look like and how can we create them?

How can we solve quadratics?

CONCEPT A	CONCEPT B	CONCEPT C
Understand quadratic patterns	Writing rules and creating graphs	Using factoring to solve
ESSENTIAL QUESTIONS A	**ESSENTIAL QUESTIONS B**	**ESSENTIAL QUESTIONS C**
What pattern is created by multiplying two linear equations? How are the zeroes, y-intercept, and max/min related to the two linear equations?	How can we use the distributive property to rewrite equations from factored form to standard form? How can we use the information in rules to create graphs of quadratic equations?	How can we rewrite equations from standard form into factored form? How can we use factoring and the zero product property to solve quadratics?
VOCABULARY A	**VOCABULARY A**	**VOCABULARY A**
Quadratic, vertex, parabola	Distributive property, FOIL, Punnett Square, y-intercept, x-intercept, opens “up” or “down”	Factoring, zero product property

ADDITIONAL INFORMATION/MATERIAL/TEXT/FILM/RESOURCES