The Parameters of Transformational Geometry

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Objective

Geometry is a math topic that is full of applications that can make it accessible to all students. Unfortunately, when we follow a textbook, we take away much of the excitement that could draw them in. Geometric transformations are found in nature – flowers, butterflies, and reflections in still lakes. There are transformations in music, in art, even in car hubcaps. Armed with a greater appreciation and understanding of transformations, my objective is to have students apply knowledge of geometric transformations to what they see around them.

The NCTM Geometry Standard states that instructional programs should enable all students to apply transformations and use symmetry to analyze mathematical situations. In grades 9-12 all students should understand and represent translations, reflections, rotations, and dilations of objects in the plane by using sketches, coordinates…and use various representations to help understand the effects of simple transformations and their compositions. The Delaware 10th grade GLE (Grade Level Expectation) states that students should be able to determine the results of multiple transformations and determine the transformations required to obtain the finished product from the original shape. These are the primary standards being addressed in this curriculum unit.

I teach in the New Castle County Vocational-Technical School District. All four high schools in the district use the Core Plus integrated math curriculum. Students have four 90-minute block periods per semester; ninth graders take two semesters of math, while upperclassmen take one semester per year. The majority of ninth graders begin in Course 1; however, the lowest students are grouped together and work at a slower pace, while the highest-level students are grouped together and begin in Course 2. Students are grouped heterogeneously after ninth grade. All students complete Course 3 before they have the option of continuing in Course 4 or going to Intermediate Algebra and then Precalculus – both traditional math courses. Each integrated math course covers four or five distinct textbook units per semester. The first Geometry unit is taught in Course 2, focusing primarily on Pythagorean Theorem and surface area and volume of three-dimensional solids. I am writing this curriculum unit to enhance the coordinate geometry unit in Course 3. Following the content in the textbook, students begin by drawing triangles and quadrilaterals in a coordinate plane, then develop the distance and midpoint formulas and use measurements to identify triangles or quadrilaterals based on parallel, perpendicular and congruent sides. My colleagues and I have developed these lessons so that students master the necessary skills. The second lesson of the unit focuses on transformations:

translations, reflections, rotations and dilations, and composite functions. The textbook unit is superficial as are the expectations on our (district) assessments. Therefore, the topic for this curriculum unit will be Transformational Geometry to extend student understanding so they may apply it to real-world applications.

In my early readings, I found that the way I used to define the parameters in transformations limits my students' deep conceptual understanding. As a district, we only expose students to four reflections (across the two axes, $y = x$, and $y = -x$ lines), three counterclockwise rotations (90°, 180° and 270°) only centered at the origin, and translations defined in terms of their horizontal and vertical movement. Students do experiment to discover the rules that move objects in a prescribed manner, but are not expected to remember them since we provide the transformation rules on a Reference Sheet for all assessments. Some specific concepts that are weak or missing from my current presentation of geometric transformations that I plan to address in this unit are 1) a reflection line can be anywhere in the plane and is the perpendicular bisector of the line connecting a pre-image point to its image, 2) the center of a rotation can be anywhere (not only the origin) and is equidistant from each pre-image and corresponding image point, and 3) the composition of transformations can be defined as a single (often different type of) transformation. I believe that students will not need to depend on rules being provided to them if they have more opportunity to manipulate the parameters of geometric transformations and their effects on the pre-image. My goal for this unit is to have students understand how the parameters of transformations affect the image created in terms of position, orientation and size.

Background

The Geometry we know began with Euclid about 300 years B.C.E. He compiled the work of many mathematicians into the well-known text called the *Elements*. In it, Euclid began with definitions and proved propositions. As we learned in our seminar, Euclid used constructions in his proofs, and did not accept any claims that could not be proven with constructions. For that reason, Euclid did not formally include transformations in the *Elements*.

At the risk of dating myself, I never studied geometric transformations in school. My Geometry course consisted of formal two-column proofs to prove triangle congruence and properties of segments related to circles. I have a vivid recollection from my most confusing/thought-provoking Geometry class where we had to state whether a condition (a given statement) was necessary and/or sufficient to determine whether a figure was a parallelogram, rectangle, rhombus, square, etc. Beyond that, I don't remember all that was included in the course. As a result, geometric transformations were a new topic for me, so I "took the easy way out" and followed the textbook and district curriculum requirements. After participating in this seminar, I have a much stronger understanding of transformations and how to connect them to concepts and applications my students already know.

The first thing I learned was that transformations on points are directly related to function operations: addition, subtraction, multiplication, division, and composition. In Geometry, a transformation is called a Mapping, or how an input point (called the pre-image in the Core Plus textbook) "moves" to another output point (called the image). As in composition of functions, composition of transformations is a sequence of two mappings, using the output of the first as the input for the second.

I also learned that some of the nine rules that apply to operations on the set of Real Numbers apply to functions, and also to transformations. There are four rules for addition: Commutative Property of Addition ($a + b = b + a$), Associative Property of Addition ($a + (b + c) = (a + b) + c$), Additive Identity ($a + 0 = a$), and Additive Inverse ($a + -a = 0$). There are also four rules for multiplication: Commutative Property of Multiplication ($a \cdot b = b \cdot a$), Associative Property of Multiplication ($a \cdot (b \cdot c) = (a \cdot b) \cdot c$), Multiplicative Identity ($a \cdot 1 = a$), and Multiplicative Inverse ($a \cdot \frac{1}{a} = 1$, where a is not the additive unit). The ninth rule is the Distributive Property of Multiplication over Addition ($a(b + c) = ab + ac$). Students should already be familiar with these rules, but can always use reinforcement, especially in different contexts.

In this unit students will study three types of Geometric Transformations: Isometries, Dilations and Composition of Transformations.

Isometries

Isometries are transformations, or movements, in the Euclidean plane that preserve the distance between points. In my textbook, they are referred to as Rigid Transformations. The transformations themselves map a point to a point. However, a line segment or a figure, such as a polygon, is a set of points and can be transformed by performing the same mapping on every point. My students recognize these rigid transformations of figures as the "same shape, same size, somewhere else" in the plane. They calculate lengths and slopes of input and output images to prove that size and shape are preserved. There are four isometries: Reflections, Translations, Rotations, and Point Inversions. The formality of these isometries is what is missing in my curriculum. Learning about these isometries had an immediate impact on how I teach geometric transformation; it deepened my understanding so I was able to teach with more confidence, and add more rigor and examples to my lessons.

Reflection

To begin, I learned that some textbooks (obviously not ours) use notation, similar to function notation, for mappings. Reflections of a point to a point, known to younger students as flips, can be denoted with $r_k(A)$, read as "the reflection of point A across line *k*." The parameter for reflections is the line of reflection; the line of reflection can be anywhere in a plane. In the case that point *B* is on the reflection line, then its reflection point remains on the line, $r_k(B) = B$. If *A* is not on line *k*, its reflection point is *A'*: $r_k(A) = A'$. In addition, the line *k* is the perpendicular bisector of the line segment connecting the pre-image to its image, *AA'*. This is a very important fact I rarely emphasized with my students prior to this year, but provides an excellent reinforcement to what they learned about slopes of perpendicular lines, using the distance formula to prove that line segments are congruent, or finding the midpoint of a line segment to prove the line of reflection intersects at the midpoint; it also reinforces vocabulary terms. Students may observe that a polygon reflected across a line reverses its orientation. This observation could even help them identify a reflection. It can be visualized by observing the labels on the vertices a polygon. If the pre-image were labeled clockwise, then its reflection image would be labeled counterclockwise, as shown in Figure 1.

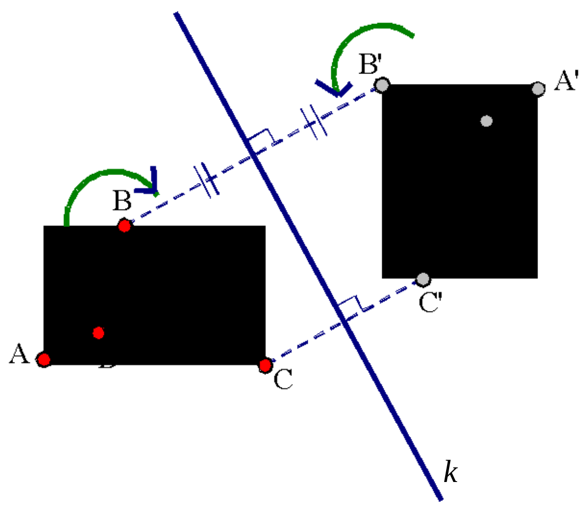


Figure 1

As an example of a composite transformation, reflecting point *A* across the same reflection line, *k*, twice yields an Identity (the output of the composition is the same as the input). Using notation, $r_k(r_k(A)) = r_k(A') = A$ or $(r_k \circ r_k)(A) = A$. While this notation is beyond the level of my Course 3 students, it is something I can use as an example for my Precalculus students to make the connection between algebra and geometry. For my Course 3 students, I can use the notation $A(x,y) \rightarrow A'(x',y') \rightarrow A(x,y)$ to denote the pre-image point, *A*, mapping to the image point, *A'*, and then back to the original coordinates of point *A*.

Translation

In the seminar, and in my readings,[i] translation is defined as movement parallel to a vector that has both length and direction. Elementary students call the movement a slide. The orientation of a translated figure is preserved (i.e. clockwise labels remain clockwise). The notation for translating point *P* parallel to vector $\overrightarrow{AB}$ is $T_{\overrightarrow{AB}}(P) = P'$. The parameter for translation is the vector, which is a line segment that defines the direction and length of the translation. An alternate notation in terms of coordinates of a point is $T_{a,b}(x,y) = (x+a, y+b)$. Each point is moved *a* units in the horizontal direction and *b* units in the vertical direction. Positive values of *a* and *b* indicate a move to the right and/or up; negative values of *a* and *b* indicate a move to the left and/or down on the coordinate plane. I teach translations as part of a coordinate geometry unit, so this second notation makes sense to my students. Most of them have not learned about vectors yet, so it will require additional instruction. However, when I incorporate activities that take them "off grid," students will need an alternate method of defining a translation. Defining translations using vectors would require them to draw parallel and congruent lines, which, again, is reinforcement of previously learned skills. Then, by connecting the endpoints of the vector $\overrightarrow{AB}$ (refer to Figure 2) to the corresponding endpoints of the line segment *PP'*, students can see and prove by slopes and lengths of line segments that a parallelogram is formed.

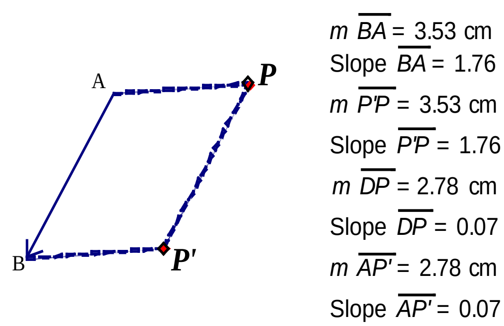


Figure 2

Rotation

Rotations in the early grades are often called turns. A positive rotation is a counterclockwise rotation and the orientation of a figure is preserved with rotation; I think this fact could help students distinguish between reflections and rotations in some situations. To describe a rotation, we need to define two parameters: the center point for the rotation, and the angle of rotation. The notation for rotating point *P* theta (θ) degrees around a center at *O* is $R_{0,\theta}(P) = Q$. If we draw a line segment from the center to point *P*, it becomes the radius of rotation. In other words, both P and Q lie on a circle with radii OP = OQ, and the measure of the central angle ∠POQ formed is theta (m∠POQ = θ). Furthermore, a composition of two rotations is Commutative, meaning the order does not change the output; it is the sum of the two angles of rotation. Using function notation, $R_{0,\theta} \circ R_{0,u} = R_{0,u} \circ R_{0,\theta} = R_{0,\theta+u}$, where θ and u are the two angle measures as illustrated in the Figure 3.

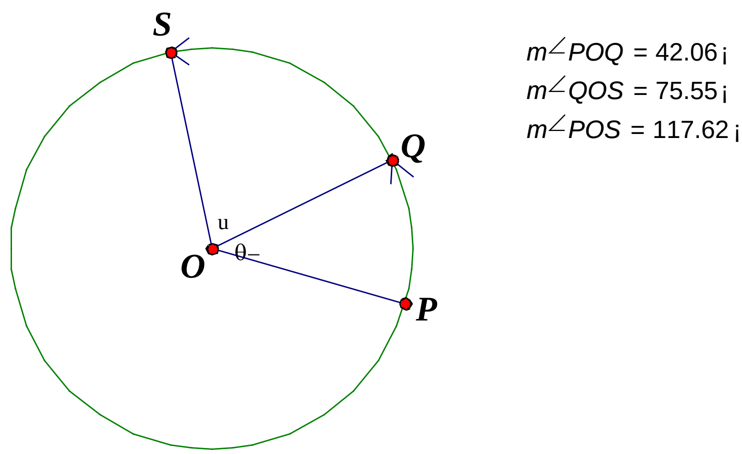


Figure 3

In my district curriculum, we only rotate around the origin (0,0), and only consider 90°, 180° and 270° rotations, which is an oversimplification of all the possibilities. We also ignore the fact that the distance from the center of the rotation to both the input and the output is equal. Students' conceptual understanding may increase if they experiment with rotating line segments first, followed by polygons rotated about different center points, both on and off the polygon itself. Computer software will be especially valuable in helping students perform many rotations in less time so they can focus on the effect of changing the center, and what rotations of varying angles look like.

Point Inversion

When drawn, a point inversion looks like a 180° rotation, or ½ turn around point *P*. My curriculum does not identify point inversions as a distinct transformation. For a point inversion, the parameter is point *P* through which to invert point *A*. *P* is then the midpoint of the line segment *AA'*, connecting the input to the output. I must admit that I never heard of a point inversion before this seminar. I have, however, taught my students about 180° rotations around the origin. When rotating around the origin, students discover that a point (x,y) maps to (-x,-y). That's fine, but I now realize there is so much more I can do with point inversions. As I tried to make sense of my own seminar notes, I selected a random point $P(2,4)$ through which to invert point $A(1,8)$. Since P is the midpoint of the line segment AA', I used the concept of slope

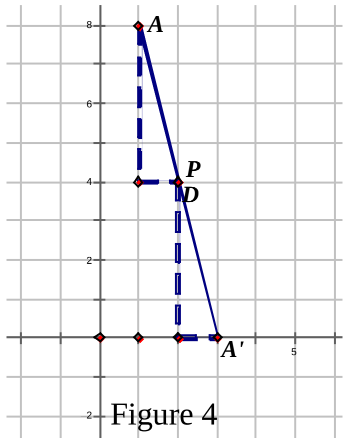


Figure 4

(in this example, the y-value decreased by 4 as x increased by 1) to determine that the coordinates of A' must be (3,0). I could use either the distance or the midpoint formula to confirm that P is indeed the midpoint of AA'. I can also use the three points on the line (or any two of them, really) to write the equation of the line as $y = -4x + 12$. Giving my students an example like this is yet another way of connecting coordinate geometry to algebra, which is exactly what an integrated math course should do!

Composition of Isometries

In seminar we learned that the output of any composition of isometries is also an isometry. Furthermore, the result of composite isometries can be defined by a different single isometry. One example, shown in Figure 5a below, is a composition in which point A is reflected across two parallel lines, k_1 and k_2, in sequence, but can be defined as a single <u>translation</u>. As proof, let B be the point of intersection of the line segment AA' with k_1, and let C be the intersection of $A'A''$ and k_2. Because k_1 is the perpendicular bisector of AA', the length of AB is congruent to BA'. Likewise, the length of $A'C$ is congruent to

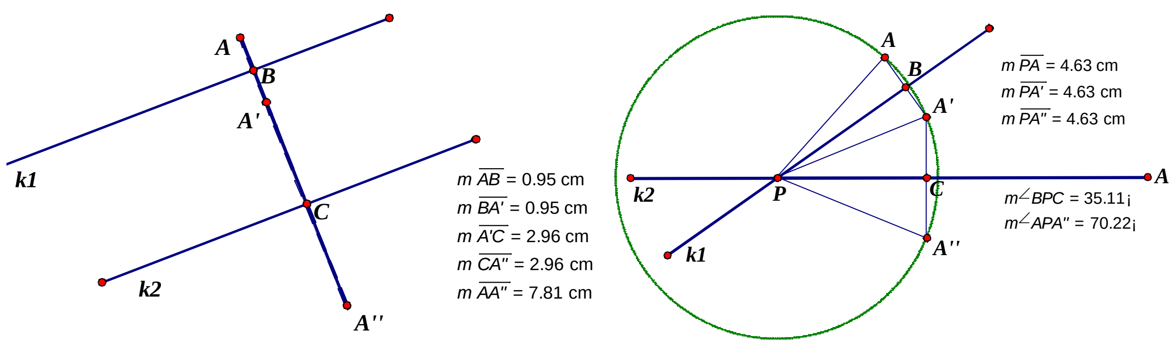


Figure 5a   Figure 5b

CA''. Therefore the sequence of two reflections across two parallel lines, $(r_{k_2} \circ r_{k_1})(A) = A''$, is equal to $T_{2\overrightarrow{BC}}(A) = A''$, a translation that is twice the distance between the two parallel lines as measure by the line segment $\overline{BC}$, and perpendicular to both lines.

A second example, shown in Figure 5b, is a composition in which point *A* is reflected across two intersecting lines, k_1 and k_2 that intersect at point *P*, in sequence. The composite reflection $\left(r_{k_2} \circ r_{k_1}\right)(A) = A''$ can be defined as a single rotation, centered at *P*, of an angle twice the angle formed between the two lines, $R_{P,2\cdot m\angle BPC}$. For proof, referring to Figure 5b above, let *B* be the intersection of k_1 and *AA'*, and let *C* be the intersection of k_2 and *A'A''*. Since k_1 is the perpendicular bisector of *AA'*, *AB* is congruent to *BA'*. Since k_2 is the perpendicular bisector of *A'A"*, *A'C* is congruent to *CA"*. In addition, ∠PBA is congruent to ∠PBA' since both are 90°. Line segment *AB* is a common side; therefore, ΔAPB ≡ ΔA'PB by the side-angle-side (SAS) triangle congruence postulate. Using the same reasoning, ΔA'PC ≡ ΔA''PC. Since corresponding parts of congruent triangles are congruent, ∠APB is congruent to ∠A'PB, and ∠A'PC is congruent to ∠A''PC. For the same reason, the lengths of *PA*, *PA'*, *PA''* are all congruent – points *A*, *A'* and *A"* are all equidistant from *P* (i.e. they lie on a circle with radius = m*PA* = m*PA'* = m*PA"*). As shown in the figure, ∠A'PB + ∠A'PC is equal to the measure of the angle between the two intersecting lines, and the transformation reflecting from *A* to *A'* to *A''* can be described by a single <u>rotation</u> with center *P* equal to twice the angle measure between the two lines.

Armed with my newfound knowledge, I returned to my Core Plus textbook lesson on combining transformations (Course 2, Unit 3, Lesson 2, Investigation 3), and gained appreciation for it. One exercise has students perform a composition of translations. Students compare size, shape and orientation of the final output image to determine that the composition is indeed an isometry. They also compare input and final output coordinates to identify the overall horizontal and vertical components of the translation. In the end, students discover that a composition of two translations can be defined by a single translation equal to the sum of the two translations: $(T_{\overrightarrow{CD}} \circ T_{\overrightarrow{AB}})(A) = T_{\overrightarrow{AB}+\overrightarrow{CD}}(A) = A'$. In addition, the composition is Commutative (changing the order of the translations does not affect the final output).

This semester, I was able to use Geometer's Sketchpad software on a classroom set of computers which allowed students to perform other composite transformations in the lesson more quickly; in previous years, it was a tedious process to draw each one on graph paper, identify coordinates and discover mapping rules. With the visual images in front of them, students were able to recognize that a 90° counterclockwise rotation about the origin, followed by a reflection across the y-axis could be described as a single reflection across the line $y = x$. Using the notation in the Core Plus textbook, the composition $(r_{y-axis} \circ R_{O,90^\circ})(A) = r_{y=x}(A)$ is $(x,y) \to (-y,x) \to (y,x)$. Students previously discovered that the rule $(x,y) \to (y,x)$ defines reflection across the line y = x. However, if the transformations were changed, the resulting image would be defined by a single reflection across the line $y = -x$: $(R_{O,90^\circ} \circ r_{y-axis})(A) = r_{y=-x}(A)$ or $(x,y) \to (-x,y) \to (-y,-x)$. Thus, the composition of reflection and rotation is not Commutative; the order of transformations affects the final output. Students confirmed the results using coordinates of the input and output, and I used class discussion to demonstrate the idea of composing algebraic functions. My students have not learned about functions formally yet, but they have enough experience to recognize and use the algebraic process involved (i.e. substitution and Distributive Property).

Dilations

Dilations are transformations that do not preserve distance. The parameters for dilations are the center of the dilation and scale factor. Dilations are called stretches, or enlargements, if the scale factor is greater than one. They are called shrinks, or contractions or reductions, if the scale factor is less than one. A dilation of point *P* from point *O* by the scale factor *k* maps the point to *P'*. The points *P*, *P'* and *O* are collinear, and the length of *OP'* is equal to the length of *OP* multiplied by *k*. The notation for dilations with scale factor, *k*, is $D_{O,k}(x, y) = (kx, ky)$. This notation clearly indicates that the output image can be obtained by multiplying both the x- and y-coordinates of the pre-image by the scale factor.

In the case of a figure being dilated by a scale factor (not equal to one), the figure maintains its shape (corresponding angles of a polygon are congruent), but differs in size. Dilations create similar figures in which every length is proportional to its corresponding length (i.e. the ratio of corresponding parts is constant). Furthermore, the distance from the center to the output, *OP'*, is equal to the distance from the center to the input, *OP*, multiplied by the scale factor.[ii] Students can practice their skills of calculating lengths and slopes in order to prove that input and output images are similar – parallel and perpendicular relationships remain intact and lengths are proportional.

In the many years that I've been teaching, I have been on a mission to make students understand that the area of a two dimensional figure enlarged or shrunk by a scale factor, *k*, changes by a factor of k^2. I can admit to making progress over the years, but, still, the intuitive response from most students is that the input area is multiplied by the scale factor alone to get the output area. The study of dilations, especially using interactive software, is yet another opportunity to emphasize the correct relationship.

Composite transformations involving dilations present additional opportunities to connect and reinforce algebraic operations, including the order of operations. For example, combining two dilations having the same center, in sequence, can be defined as a single dilation, with the same center, and overall scale factor equal to the product of the two scale factors. The composition of two dilations is commutative, as you might predict since multiplication of real numbers is also commutative. However, composition of a translation and dilation is not commutative. The order of operations comes into play since the composition involves both addition and multiplication. When the dilation comes first, followed by a translation, the horizontal and vertical components of the translation vector are only added once. For example, $(T_{2,-4} \circ D_{O,3})(x, y) = T_{2,-4}(3x, 3y) = (3x+2, 3y-4)$ describes a dilation with a scale factor of 3, followed by horizontal and vertical translations of 2 and -4 units, respectively. If the same horizontal and vertical translations were applied first, followed by the dilation, the mapping rule would be $(D_{O,3} \circ T_{2,-4})(x, y) = D_{O,3}(x+2, y-4) = (3(x+2), 3(y-4))$. Applying the Distributive Property, the mapping becomes $(3x+6, 3y-12)$. The resulting image was translated 6 units horizontally and -12 units vertically, which is three times the magnitude of the previous example. Again, the use of computer software would allow students to visualize the difference between the two composition transformations.

Applications of Transformations

According to the New York State Regents practice website, "Transformational Geometry is a method for studying geometry that illustrates congruence and similarity by the use of transformations."[iii] There are numerous applications of transformations within easy access of a student's world and/or experiences. Students can recognize congruence and symmetry in architecture, nature, logos, wallpaper, art, and even music. They can recognize similarity in scale drawings, photo enlargements, stacking cups, etc.

To determine whether an object has symmetry, consider whether reflections, rotations, or translations preserve the properties of the pre-image. For example, a square has eight symmetries: four lines of reflection – its two diagonals plus two lines, each of which are perpendicular bisectors of opposite sides, and four rotations about its center at the intersection of its diagonals – 90°, 180°, 270°, and 360°. With each symmetry, the square "looks" as though it were mapped onto itself, even though individual points were actually transformed.

Circles have an infinite number of rotational symmetries about its center. However, decorative round objects have fewer rotational symmetries. Hubcaps have multiple rotational symmetries; the number depends on its design. The logo for BP gas stations (an image of a sun on a green circle) also has numerous rotational symmetries, equal to the number of points along the circumference of the sun. The angle measures that would define the rotational symmetries are multiples of 360° divided by the number of rays of the sun.

Nature provides examples of reflection symmetry in still bodies of water, butterflies and in some flowers, such as orchids. Flowers such as daisies or sunflowers have rotational symmetry. Starfish, sand dollars and honeycombs also exhibit rotational symmetry. The coloring on snakes demonstrates translational symmetry, [iv] and footprints demonstrate a combination of translation and reflection symmetry, sometimes called glide reflections.

Studying symmetry in art can provide a connection to history and culture.[v] Ancient mosaic tile designs, such as those used in ancient baths, along with religious ritual ornaments are beautiful examples of transformational geometry used over many, many centuries. A more contemporary artist, M. C. Escher, uses symmetry in his tessellation drawings, which may interest students, as well. Still another art application of multiple symmetries is wallpaper. In fact, there are 17 different groups of symmetries that specifically define all of the possible patterns found in wallpaper.[vi]

As a final example of a transformation application, consider music. I had never thought of connecting geometry to music, but an article entitled “Listening to Geometry” in *The Mathematics Teacher*, describes translations, reflections, rotations and dilations with respect to music.[vii] I also heard an article on National Public Radio® (NPR®) refer to the symmetry of a ballet dancer’s body when standing in first position. “And the other positions simply map the directions of the body in the ways in which you might travel or move in an efficient and graceful way...”[viii]

Strategies

My intention is to put aside my Course 2 *Core-Plus Mathematics* textbook and replace the first two investigations of Unit 3, Lesson 2 “Coordinate Models of Transformations” with the activities described below. It’s not that students can’t learn about transformations from the textbook activities, but they are tedious and somewhat rote. Students must plot figures repeatedly, and if they misplace a point, or misread coordinates, they can’t find the patterns or “rules” that map a pre-image to its image. Instead, students will see transformations using premade Geometers Sketchpad (GSP) files on a classroom set of Netbook computers. I think the software will significantly help me emphasize the importance of the parameters of the transformations. For example, the textbook activities involve four reflections - across the two axes and the $y = x$ and $y = -x$ lines only, whereas the GSP software allows students to place, and move, the reflection line anywhere (no grid required). As a result, students can focus more on what a reflection looks like and the fact that the reflection line is always perpendicular to the line segment connecting corresponding input-output points. In addition, the software measures and calculates angles, lengths, midpoints, perimeter and area; students can again, focus on relationships rather than tedious and repetitive calculations for greater understanding. Once students understand the relationships, homework assignments will require them to plot the pre-image and image and calculate lengths and slopes to prove the relationships with respect to the parameters. Homework assignments will come from the textbook, so that students become familiar with the wording of questions that they will most likely see on District Unified Assessments.

Students will work in pairs as they learn to use the GSP software. Throughout the unit, partners will share and verbalize their observations. I believe my students will gain greater conceptual understanding in less time because they will see more examples simply by "clicking and dragging" points. For example, they can easily change the position of the center of rotation to see its effect on the transformation. They can rotate through any angle, not just 90°, 180° and 270° counterclockwise. Again, finding the distance from pre-image and image points to the center of rotation is only "a click away" so students see that it remains constant without the calculations interfering with the understanding. We will also have full class discussions to summarize and formalize the results of students' investigations. During class discussion, I will formalize the transformations using the notation I used in the Background section above; however, I will also write the same transformations in terms of coordinates because that is the form students will see on district-made assessments. Throughout the unit, I will take advantage of GSP premade activities that extend students' thinking for those that complete the primary assignments more quickly.

The activities in this unit will lead students through the five levels of Bloom's Taxonomy. At Level I: Knowledge, students will take a pretest in which they will match elementary and formal names of transformations to assess their vocabulary level. Students perform transformations in lower grades, but sometimes call them by different names (i.e. slides, flips and turns vs. translations, reflections and rotations). The pretest is also designed to evaluate whether students have any misconceptions that need to be corrected. To move students to Level II: Comprehension, I will have students compare inputs (pre-images) and outputs (images) visually, as well as by coordinates and calculated measurements of distance, slope, perimeter and area. They will change parameters, look for patterns and develop rules that define the different types of transformations. At Level III: Application, students will apply what they learned about single transformations to composite transformations. They will perform two transformations in sequence, then develop a single rule that maps the pre-image to its final image and use GSP or plotted points to confirm their conjectures. They will also make and test conjectures about whether the order in which two transformations are performed affects the output. To demonstrate they have reached Level IV: Analysis, students will construct the parameters of a transformation, given a pre-image and image. For example, given a pre-image and its reflection image, students will construct the reflection line and justify that it is the perpendicular bisector of the line segments connecting corresponding input-output points using appropriate length and angle measurements. At Level V: Synthesis, as a final assessment, students will create a display (art or application) demonstrating specified transformations and justify how the transformations were used. This assessment will serve as a post-test to demonstrate that students have in fact reached a higher level of knowledge.

In alignment with NCTM and Delaware Process Standards and the Mathematical Practices of the Common Core State Standards, through class discussion, I will connect the patterns and rules for geometric transformations to algebraic properties students have studied previously. In addition, as much as possible, I want students to relate the transformations they learn to contexts related to their career areas or hobbies.

Activities

Pretest

The first activity at the start of this unit will be a pretest. A copy of the pretest can be found in Appendix B. The purpose of the first multi-part question is to assess students' vocabulary by matching elementary and formal terms for transformations using a Word Bank. The remaining questions ask students to perform basic transformations and explain their methods so I can determine if they have misconceptions. The first transformation is to reflect a flag across a line that is drawn. I will be able to determine whether they understand that each point of the input corresponds to a point that is equidistant from the reflection line, but on the opposite side of it. I am also curious to see if any students actually fold the paper in order to draw the mirror image of the flag, which would demonstrate understanding of the concept of reflection. The next two transformations are rotations – one has its center on the flag, the other has its center at a point not on the flag. (I expect students to have more difficulty with the second rotation.) I will observe students as they work on these problems, especially to see if any of them rotate their papers to help them visualize the rotation. The 4th transformation is a translation specifying positive and negative components in the horizontal and vertical directions on a coordinate grid. The purpose of the positive and negative components is to determine whether students understand directionality of positive and negative numbers with respect to a coordinate grid. The final transformation is a dilation on a coordinate grid with a scale factor of 1/3. I will get a sense for whether or not students relate the given scale factor to coordinates by where they position the smaller image on the grid. They may understand that the scale factor changes length, but may not apply the scale factor to individual x- and y-coordinates. I will immediately review the vocabulary section of the pretest with the class to formalize the vocabulary of transformations: translations, reflections, rotations, and dilations.

Investigations

Using the terminology of the *Core-Plus Mathematics* textbook, students will perform investigations to learn about the parameters of transformations, and how to follow and write rules that map points from a pre-image to its image. I found two premade GSP files online that I will use as the basis for these investigations. The first can be downloaded from http://sketchexchange.keypress.com/browse/topic/secondary-school/by-recent/20/96/transformations-in-sketchpad. It gives step-by-step instructions for each transformation, along with some practice. "Poly" is a bird that is used as the pre-image throughout the file. As practice, students are prompted to move Poly in a prescribed way, mostly by trial and error. For the following discussion, I will refer to this file as the *Keypress* file.

The second premade file I will use extensively can be downloaded from http://www.mathbits.com/MathBits/GSP/Transformations.htm. The pages in this file focus more on coordinates so students can begin to recognize patterns to describe movements. Each image is color-coded which makes it easy to determine which image is which and helps when comparing them. There are downloadable worksheets that go along with the GSP file that tell students to write and test rules for basic mappings, and consider whether lengths are preserved under the transformations. For the following discussion, I will refer to this file as the *MathBits* file.

Both files have multiple pages for all the different transformations. I would prefer to focus on one transformation at a time, so I will break the files into smaller files, as described below, and save them for student access from the school's public drive.

Translations

The first page of the *Keypress* file gives step-by-step instructions to translate points by vertical and horizontal distances and also by vectors. Once students learn the steps to perform translations, the second page of the file shows a map of a popular amusement park. Students gain experience using the Transform menu on the software toolbar as they translate Poly from one position to another on the map. I will save just these two pages as one file titled Translate1.

After this initial introduction to GSP and translation, students will open the Translate2 file page that I saved from the *MathBits* file. This page shows horizontal, vertical and diagonal translations of a triangle. It provides coordinates and lengths for each vertex and side, respectively, of the pre-image and images. Students will complete the worksheets that go along with this page. At the completion of this activity, I expect students to be able to move a pre-image to the right by adding a positive number to the x-coordinate, and left by adding a negative number to the x-coordinate. Likewise, they will be able to move a pre-image up by adding a positive number to the y-coordinate, down by adding a negative number to the y-coordinate, and diagonally by adding to both the x- and y-coordinates. This understanding is sufficient for our district assessments. However, I would like to extend my students' understanding of translations. One additional point I want to make with my students, that is not included on the *MathBits* worksheets, is that the orientation of the pre-image is maintained in the image. In other words, Poly is always facing the same direction through all translations, or the order of the labels on vertices remains the same – clockwise or counterclockwise.

The previous two activities depend on translating distances relative to a coordinate grid. To extend students' understanding, I adapted an activity from *Exploring Geometry with The Geometer's Sketchpad*. My written instructions are in Appendix C. This investigation focuses on translations using vectors. By initiating the vector at the origin (0,0), I think students will easily see the connection between vectors and horizontal/vertical movements. Furthermore, GSP allows them to move the end of the vector to different lengths and horizontal/vertical positions and instantly see its effect on the translation. Students will also connect points to form a parallelogram, as described in the Background section (refer to Figure 2), and measure slopes and lengths to prove the quadrilateral formed is indeed a parallelogram - reinforcement of the first lesson of the textbook unit.

Reflections

There are three pages for Reflections in the *Keypress* file that I will save as a single file titled Refect1. The first one gives step-by-step instructions for reflecting Poly across a line segment. Students can drag an endpoint of the reflection line to see that the distance from Poly to the line and from the line to Poly's image remains the same. I will actually add instructions to this page to direct students to construct a line segment connecting Poly's nose to her reflection's nose. Next, they will use the Construct menu to mark the intersection of the two line segments. Students can then use the Measure menu to find the distance between Poly's nose and the intersection point and the intersection point and Poly's reflection's nose. They can also measure the angles formed between Poly's nose and the reflection line. Then by dragging the endpoint of the reflection line, students will observe that the angle measures a constant 90° and, while the distance to the intersection point changes, the distances remain equal to each other no matter where the line is. This will be my opportunity to emphasize that the reflection line is the perpendicular bisector of the line segment connecting corresponding points of a pre-image and image.

The second Reflection page of the *Keypress* file provides practice with the GSP procedure for reflection. It directs students to draw a reflection line (it represents a window) and reflect Poly across it. Then, they can drag the reflection line until Poly's image aligns with the image already on the screen. The third page is actually a composite transformation – reflecting Poly's name and then their own across two parallel lines in sequence. I will use this page for students that finish early.

While the *Keypress* file allows students to see that reflection can occur across any line, anywhere in a plane, the *MathBits* file demonstrates reflections across the same four lines used in the *Core-Plus Mathematics* textbook. The *MathBits* file and associated worksheets focus on the changes in coordinates through reflections, writing rules to map a point to its reflection, and measuring lengths of sides of the pre-image and image to prove that the lengths are preserved in a reflection; thus, reflection is an isometry. I will save this single page in a file named Reflect2, and then create two additional pages to have students reflect across the $y = x$ line on one, and the $y = -x$ on the other. On both pages, students will construct a polygon to reflect, and draw line segments connecting corresponding pre-image and image points. They will then measure the distance from both the pre-image and image to the reflection line, as well as the angle formed at the intersection of the line segment connecting pre-image to image and the reflection line. The purpose of these additional pages is three-fold: to reinforce the GSP procedures for reflections, to perform constructions, and prove relationships based on measurements. As part of the discussion about reflections, I will make the point that the orientation of the image is different from the pre-image; Poly's image is facing the other direction, and labels on vertices of a polygon go in the reverse direction on either side of the reflection line.

The homework assignment to reinforce reflections will require students to plot a triangle, reflect it across a given line, calculate the lengths of all three sides of the image and the pre-image. Students will also draw line segments connecting corresponding vertices, measure their slopes and calculate their midpoints to prove that the reflection line is indeed the perpendicular bisector of each segment.

Rotations

As I did for the previous two types of transformations, I will save two pages from the *Keypress* file as Rotate1 and two pages from the *MathBits* file as Rotate2. The first page of the saved *Keypress* file is the only one devoted solely to rotations. It gives step-by-step instructions for rotating through a fixed angle (specified by the user) and rotating through a marked angle. Students practice the procedure as they "swing" Poly from one branch of a tree to another. The second page provides practice of all three isometries; students use translations, reflections and rotations to move Poly through an obstacle course…fun!

The first page in the saved *MathBits* file demonstrates the three basic rotations (90°, 180° and 270° counterclockwise), and all coordinates of the pre-image and image are shown. The second page instructs students to vary the angle of rotation to any angle and also to "Move the Center of Rotation point for further investigations." In my opinion, there are two negatives about the associated worksheets for the *MathBits* file. First, the notation used, $R_{270°}(x, y)$, does not specify the center of rotation, so I will either edit the worksheet before distributing it, or mention to my students that we are using the origin as the center of all rotations, unless specified otherwise. Second, the worksheets do not direct students to measure the distance from both the pre-image and image points to the center of rotation, something I have learned is critical in understanding rotations. Therefore, I will add instructions to the worksheets directing students to construct line segments from pre-image and image points to the center of rotation and measure their lengths. I will also include instructions to construct a circle using the center of rotation and radius equal to one of the line segments drawn. Then, students can vary the angle of rotation and clearly see that with each rotation, the corresponding image points lie on the same circle, and therefore, are equidistant from the center of rotation. Finally, for consistency, I will include discussion about the orientation of the image being the same as the pre-image through rotations.

Dilations

Only the *MathBits* file has examples of dilations, and associated worksheets, so I will save the single page in a file named Dilate. The dilated triangles are examples of scale factors both greater than and less than one. The worksheets address changes in coordinates, as well as changes in the lengths of sides of the triangles. One critical concept that I feel is missing in these worksheets is the effect of dilations on area and perimeter, so I will need to provide additional instructions. GSP performs the calculations, which will make it simple for students to collect data that can help them discover that the perimeter of a polygon image, being a one-dimensional measurement, varies directly with the scale factor, whereas its area, being a two-dimensional measurement, varies with the square of the scale factor. Another missing concept in this file is the center of dilation; the notation on the worksheets assumes the center is the origin. I will instruct students to mark a different center for the dilation, and perform multiple dilations with a range of scale factors. Then, I will have them construct a line through all corresponding pre-image and image points. After constructing these lines for more than one set of corresponding vertices and/or dragging to change the position of the vertices, I think students will be fascinated to see that all of the lines intersect at the center of the dilation.

Composite Transformations

At this point in the unit, I think students will be comfortable enough with the GSP software to return to the textbook activities, at least with some modifications to the directions. I will, however, give students the option of using GSP or graph paper to draw the associated figures and transformations. In general, I think the 3rd investigation of Unit 3, Lesson 2 presents composite transformations in a well-organized format, leading students through the key ideas, highlighting the key transformation parameters, and writing rules to map the pre-image to the final image in a single transformation.

Proof

As a culminating, in-class activity, I will give students, working individually, three independent coordinate grids. Each grid will contain an image and a pre-image under different transformations: reflection, rotation, and dilation. Rulers, protractors, and compasses will be available. For the reflection, students will need to draw the reflection line and justify its placement based on what they learned in the unit (i.e. prove it's a perpendicular bisector). For the rotation, students will need to locate the center of rotation and justify its position (i.e. the center of a circle containing corresponding parts of both images). For the dilation, students will need to locate the center of the dilation and identify the scale factor, justifying each. Once individuals have had time to complete the task, students will join with a partner and explain their thought processes and justification in locating the parameters. Partners will provide feedback to each other, and seek experts (classmate or teacher, if necessary) to resolve any differences they have and/or clear up confusion.

Final Assessment: Project

As a final assessment that will serve as a post-test for the unit, students will create a display that demonstrates their understanding of the parameters for four types of transformations: Reflection, Translation, Rotation and Dilation. It is my hope that students will use images related to either their career areas or hobbies. The specific requirements for the project are given in Appendix D.

Appendix A – Implementing District Standards/Delaware State Standards

Content Standard 3 – Geometric Reasoning: Students will develop Geometric Reasoning and an understanding of Geometry and Measurement by solving problems in which there is a need to recognize, construct, transform, analyze properties of, and discover relationships among geometric figures; and to measure to a required degree of accuracy by selecting appropriate tools and units.

Within the Content Standard, this unit addresses the 10th grade Delaware Geometry Grade Level Expectations (GLE's) for "Location and transformation." In terms of the Common Core State Standards (CCSS) for High School Geometry, transformations are an integral part of understanding congruence (rotation, reflection, or translation) and similarity (dilation) of geometric figures.

In addition, this unit is designed to engage students in the CCSS Standards for Mathematical Practice as well as the Delaware Process Standards listed below:

Standard 5 – Problem Solving: Students will develop their Problem Solving ability by engaging in developmentally appropriate problem-solving opportunities in which there is a need to use various approaches to investigate and understand mathematical concepts.

Standard 6 – Reasoning and Proof: Students will develop their Reasoning and Proof ability by solving problems in which there is a need to investigate significant mathematical ideas…to justify their thinking; to reinforce and extend their logical reasoning abilities; to reflect on and clarify their own thinking; to ask questions to extend their thinking; and to construct their own learning.

Standard 7 – Communication: Students will develop their mathematical Communication ability by solving problems in which there is a need to obtain information from the real world through reading, listening and observing; to translate this information into mathematical language and symbols; to process this information mathematically; and to present results in written, oral, and visual formats.

Standard 8 – Connections: Students will develop mathematical Connections by solving problems in which there is a need to view mathematics as an integrated whole and to integrate mathematics with other disciplines, while allowing the flexibility to approach problems, from within and outside mathematics, in a variety of ways.

Appendix B - PRETEST – GEOMETRIC TRANSFORMATIONS

<u>Word Bank</u>: flip, slide, turn, enlarge/reduce, stretch/shrink

1. What is another name for each of the following transformations?

    a. Rotation ____________________

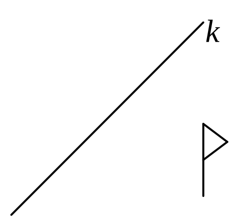
    b. Translation __________________

    c. Reflection ___________________

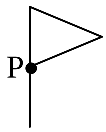
    d. Dilation _____________________

2. Flip the flag across line *k*. Tell how you decided where/how to draw the new flag.

Explanation:

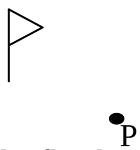


3. Turn the flag 90° counterclockwise around point P. Tell how you decided where/how to draw the new flag.



Explanation:

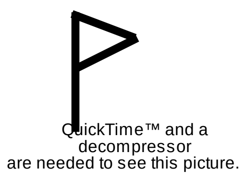
4. Turn the flag 180° around point P. Tell how you decided where/how to draw the new flag.



Explanation:

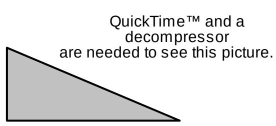
5. Slide the flag horizontally +3 units and vertically -6 units. Tell how you decided where/how to draw the new flag.

Explanation:



6. Shrink the triangle by a scale factor of 1/3, centered at the origin. Tell how you decided where/how to draw the new triangle.

Explanation:



Appendix C - Translations Using Vectors in Geometer's Sketchpad[ix]

1. In the Graph Menu, choose *Show Grid,* then *Snap Points*.

2. Draw a segment from the origin (label it A) to anywhere on the grid (label it B).

3. Highlight point B, then, in the Measure Menu, choose *Coordinates.*

4. Construct a polygon anywhere on the grid.

5. Mark the vector AB:

    a. Highlight point A and B in that order.
    b. In the Transform Menu, choose *Mark Vector*.
    c. Highlight the polygon (vertices and interior).
    d. In the Transform Menu, choose *Translate*.

6. Measure the coordinates of all vertices of the pre-image <u>and</u> image.

7. Drag point B. Look for a relationship between the coordinates of each point and its image.

QUESTION #1: Where can you place point B so that the pre-image points and image points have the same y-coordinates but different x-coordinates?

QUESTION #2: Where can you place point B so that the pre-image points and image points have the same x-coordinates but different y-coordinates?

8. Drag any of the pre-image vertices. Look for a relationship between the coordinates of each point and its image.

QUESTION #3: If the coordinates of point A are (0,0) and point B are (*a*, *b*), what are the coordinates of a point (x, y) translated by vector AB?

9. Highlight two corresponding vertices of the pre-image and image of the polygon. In the Construct Menu, choose *Line Segment*. Repeat this procedure to connect the pre-image vertex to point A and the image vertex to point B.

QUESTION #4: What type of quadrilateral is formed? Justify your answer by measuring appropriate lengths and slopes. What do these measurements tell you about translations with respect to a vector?

Appendix D - <u>Transformations - Final Project</u>

Requirements:

1. Create a display containing four pictures:
    a. One picture must show an example of Reflection.
    b. One picture must show an example of Translation.
    c. One picture must show an example of Rotation.
    d. One picture must show an example of Dilation.

    Pictures may show more than one transformation, but four different pictures are required. (For example, if a picture shows Reflection and Translation, select Reflection and then use a second picture to show Translation, or vice versa.)

    Pictures may be photos, artwork, etc. If possible, select pictures related to your career area or a hobby. (Be sure to cite your sources.)

2. Label each picture with the type of transformation illustrated.

3. Write a rule for each picture that maps the pre-image to its image.
    Use proper notation for the rule, indicating parameters of the transformation.

4. Add the location of parameters of the transformation to each picture (i.e. draw the reflection line, center of dilation, etc.).

5. Include all measurements to justify your placement of the parameters.

    You can add the measurements directly to the picture, or identify points on the picture and show the measurements elsewhere in the display.

Resources

Abney, Angel, Andy Tyminski, and Pawel Nazarewicz. "Symmetry in Nature." Jim Wilson's Home Page. http://jwilson.coe.uga.edu/EMT668/EMAT6680.2002.Fall/Nazarewicz/7210_final_2/7210_Project/index.html (accessed January 3, 2012).
Examples of symmetry in nature; great pictures.

Battista, Michael T.. "The Development of Geometric and Spatial Thinking." In *Second Handbook of Research on Mathematics Teaching and Learning*. Charlotte: NCTM, 2007. 843-908.

Bennett, Dan. "Chapter 2: Transformations, Symmetry, and Tessellations." In *Exploring Geometry with The geometer's Sketchpad Version 5*. Emeryville, CA: Key Curriculum

Press, 2012. 49-79.
Step-by-step instructions for using Geometer's Sketchpad for transformations, including activities and questions for students.

Bucher, Christopher J., and Michael Todd Edwards. "Deepening Understanding of Transformation through Proof." *Mathematics Teacher*, May 2011.
Examples of constructions and proof to extend student thinking.

Cooper, Brett D., and Rita Barger. "Listening to Geometry." *Mathematics Teacher*, September 2009.
Geometric transformations found in music.

Hollebrands, Karen . "High School Students' Intuitive Understandings of Geometric Transformations." *Mathematics Teacher* 97, no. 3 (2004): 207-214.
Examples of student misunderstandings with respect to the parameters of transformations.

Joyce, David. "Wallpaper Groups." Clark University | One of 40 "Colleges that Change Lives". http://www.clarku.edu/~djoyce/wallpaper/ (accessed January 3, 2012).
Explanation, including pictures, of all possible symmetries, all of which are found in wallpaper.

Joyce, David . "Euclid." Euclid's Elements. cs.clarku.edu/~djoyce/java/elements/Euclid.html (accessed January 3, 2012).
Version of Euclid's Elements, including applets and explanations.

National Public Radio. "Tracing Ballet's Cultural History Over 400 Years : NPR." NPR : National Public Radio : News & Analysis, World, US, Music & Arts : NPR. http://www.npr.org/templates/transcript/transcript.php?storyId=143775428 (accessed January 3, 2012).
Connection of geometric transformations to dance.

Natsoulas, Anthula. "Group Symmetries Connect Art and History with Mathematics." *Mathematics Teacher* 93, no. 5 (2000): 364-370.
Connections between ancient art and symmetry.

"Regents Prep Center GEOMETRY." Oswego City School District Regents Exam Prep Center. http://www.regentsprep.org/Regents/math/geometry/math-GEOMETRY.htm (accessed January 3, 2012).
Examples, sample questions and activities for transformations. Also, uses function notation for transformations.

"Sketch Exchange - Transformations in Sketchpad." Sketch Exchange.

http://sketchexchange.keypress.com/browse/topic/secondary-school/by-recent/20/96/transformations-in-sketchpad (accessed January 3, 2012).
A premade GSP file that teaches students how to perform transformations in GSP by moving "Poly" the bird by trial and error.

"Working with Geometer's Sketchpad - Transformations." Math Bits High School Math Resources. http://www.mathbits.com/MathBits/GSP/Transformations.htm (accessed January 3, 2012).
Source of downloadable GSP file for students to observe animated transformations and changes in coordinates. Worksheets can also be downloaded for students to work with the file.

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[i] Karen F. Hollebrands, "High School Students' Intuitive Understandings of Geometric Transformations," The Mathematics Teacher, 97(2004): 207-214.

[ii] "Regents Prep Center GEOMETRY," Oswego City School District Regents Exam Prep Center, accessed November 8, 2011, http://www.regentsprep.org/Regents/math/geometry/GT3/Ldilate.htm.

[iii] "Regents Prep Center GEOMETRY," Oswego City School District Regents Exam Prep Center, accessed November 8, 2011, http://www.regentsprep.org/Regents/math/geometry/GT0/transvocab.htm.

[iv] "Symmetry in Nature," Jim Wilson's Home Page, accessed November 11, 2011, http://jwilson.coe.uga.edu/EMT668/EMAT6680.2002.Fall/Nazarewicz/7210_final_2/7210_Project/index.html.

[v] Anthula Natsoulas, "Group Symmetries Connect Art and History with Mathematics," The Mathematics Teacher, 93(2000): 364-370.

[vi] David E. Joyce, "Wallpaper Groups," accessed October 3, 2011, http://www.clarku.edu/~djoyce/wallpaper/.

[vii] Brett D. Cooper and Rita Barger, "Listening to Geometry," The Mathematics Tea*cher*, 103(2009):108-115.

[viii] "Tracing Ballet's Cultural History Over 400 Years," accessed December 18, 2011, http://www.npr.org/templates/transcript/transcript.php?storyId=143775428.

[ix] Dan Bennett, "Translations in the Coordinate Plane" in Exploring Geometry with The Geometer's Sketchpad, (Emeryville, CA: Key Curriculum Press, 2012), 61.

Curriculum Unit Title: The Parameters of Transformational Geometry

Author: Nancy Rudolph

KEY LEARNING, ENDURING UNDERSTANDING, ETC.

Geometric transformations are functions that take points in the plane as inputs and give (map) other points as outputs. Isometries (rigid motions – translations, reflections, rotations) are transformations that preserve distance and angle, thereby producing congruent figures. Similarity transformations (dilations) produce similar figures. Rigid motions produce congruent ones, thereby formalizing the similarity ideas of "same shape" and "scale factor" developed in the middle grades. Technology can enable us to visualize the results of varying assumptions, explore various cases and consequences, and compare predictions with data.

ESSENTIAL QUESTION(S) for the UNIT

How can transformations be described mathematically? How do the parameters of transformations affect the movement of a figure in a plane? How can composite transformations (multiple transformations in sequence) be described as a single transformation? How do the parameters of transformations on figures in a plane predict the effect on length, slope, perimeter, and area?

CONCEPT A	CONCEPT B	CONCEPT C
Isometries (Rigid Transformations)	Dilations (Size Transformations)	Composite (multiple) Transformations
ESSENTIAL QUESTIONS A	**ESSENTIAL QUESTIONS B**	**ESSENTIAL QUESTIONS C**
How do the parameters of rigid transformations define the movement of a figure in a plane? How do the parameters of rigid transformations affect coordinates, and size of figures in a plane?	How do the parameters of dilations define the movement of a figure in a plane? How do the parameters of dilations affect coordinates, and size of figures in a plane?	How can composite transformations (multiple transformations in sequence) be described as a single transformation? How does the order of transformations affect the final output?
VOCABULARY A	**VOCABULARY B**	**VOCABULARY C**
Isometry, Pre-image, Image, Parameter, Transformation, Translation, Vector, Reflection, Perpendicular, Line Segment, Bisector, Congruent, Rotation, Orientation	Dilation, Scale Factor, Similar, Proportional	Composite Transformation, Commutative

ADDITIONAL INFORMATION/MATERIAL/TEXT/FILM/RESOURCES

Core Plus Mathematics textbook, Unit 3, Lesson 2

Geometer’s Sketchpad (GSP) software, preferably available on a classroom set of laptops or in a computer lab